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FINITE QUATERNION-FREE 2-GROUPS

BY

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ABSTRACT

We present an elementary proof of the classification theorem for finite nonmodular quaternion-free 2-groups. This proof does not involve the structure theory of powerful 2-groups. Such a new proof is also necessary, since there are several gaps in the original proof given in [5].

1. Introduction and preliminary results

Finite modular Q_8-free 2-groups are classified in [2]. Here we classify finite nonmodular Q_8-free 2-groups. The original proof of the corresponding classification theorem given in [5] depends on the structure theory of powerful 2-groups. Unfortunately, there seem to be some gaps in the proof of Lemmas 10 and 13 in [5]. However, it is a merit of B. Wilkens to discover a possibility for the existence of such a strong theorem. Our new proof of the classification theorem is completely elementary and does not involve powerful 2-groups. Nevertheless, the proof is very involved.

We first prove some easy preliminary results. Then we state the Main Theorem 1.7 and afterwards we describe in great detail the groups appearing in the Main Theorem. Propositions 1.8 to 1.11 describing these groups are also of independent interest, since they are needed by applying the Main Theorem in future investigations. After that a proof of the Main Theorem follows.

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LEMMA 1.1: *In a Q_8-free 2-group X there are no elements x, y with $o(x) = 2^k > 2$ and $o(y) = 4$ so that $x^y = x^{-1}$. If $D \le X$ and $D \cong D_8$, then $C_X(D)$ is elementary abelian.*

Proof: If $y^2 = x^{2^{k-1}}$, then $\langle x, y\rangle \cong Q_{2^{k+1}}$. If $\langle x\rangle \cap \langle y\rangle = \{1\}$, then $\langle x, y\rangle / \langle x^{2^{k-1}} y^2\rangle \cong Q_{2^{k+1}}$. Suppose $D \le X$, where

$$D = \langle a, t | a^4 = t^2 = 1, a^t = a^{-1}\rangle.$$

If v is an element of order 4 in $C_X(D)$, then $o(tv) = 4$ and tv inverts a, a contradiction. Hence $C_X(D)$ must be elementary abelian. ∎

LEMMA 1.2: *Let X be a Q_8-free 2-group with elements a and b of order 4 such that $[a, b^2] = [a^2, b] = 1$. If $[a, b] \ne 1$, then $\langle a, b\rangle$ is minimal nonabelian nonmetacyclic of order 2^4 and therefore $[a, b] = a^2b^2$ and ab is an involution.*

Proof: We have $\langle a^2, b^2\rangle \le Z(\langle a, b\rangle)$. Set $[a, b] = c$ and assume that $c \ne 1$. We compute

$$1 = [a^2, b] = [a, b]^a[a, b] = c^a c \quad \text{and} \quad 1 = [a, b^2] = [a, b][a, b]^b = cc^b.$$

By Lemma 1.1, c must be an involution and, by the above, $[c, a] = [c, b] = 1$. Hence $\langle c\rangle$ is normal in $\langle a, b\rangle$ and $\langle a, b\rangle/\langle c\rangle$ is abelian. Hence $\langle a, b\rangle' = \langle c\rangle$ and so $\langle a, b\rangle$ (being two-generated and with the commutator group of order 2) is minimal nonabelian (and so of class 2) and therefore $\exp(\langle a, b\rangle) = 4$. By assumption, $\langle a, b\rangle \not\cong Q_8$ and $\langle a, b\rangle \not\cong D_8$ since D_8 does not possess two non-commuting elements a and b of order 4. We have proved that $|\langle a, b\rangle| \ge 2^4$.

On the other hand, $c = [a, b]$, a^2, and b^2 are central involutions in $\langle a, b\rangle$. Set $V = \langle a^2c, b^2c\rangle$ so that $V \le Z(\langle a, b\rangle)$. We consider $\langle a, b\rangle/V$ and compute

$$a^b = a[a, b] = ac = a^{-1}(a^2c), \quad b^a = b[b, a] = bc = b^{-1}(b^2c).$$

Since $\langle a, b\rangle/V$ is Q_8-free, Lemma 1.1 implies that at least one of a^2 or b^2 is contained in V. Hence $a^2 = b^2c$ or $b^2 = a^2c$ and so in any case $c = a^2b^2$. We see that

$$(ab)^2 = a^2b^2[b, a] = a^2b^2c = 1.$$

Hence $\langle a, b\rangle = \langle a, ab\rangle$, where $o(a) = 4$ and $o(ab) = 2$. Since $|\langle a, b\rangle| \ge 2^4$, we must have $|\langle a, b\rangle| = 2^4$ and so $\langle a, b\rangle$ is minimal nonabelian nonmetacyclic of order 2^4. ∎

LEMMA 1.3 (see [4, Proposition 2.4]): *Let G be a minimal nonmodular 2-group which is Q_8-free. Then either $G \cong D_8$ or G has a normal elementary abelian subgroup $E = \Omega_1(G) = \langle n, z, t\rangle$ of order 8 with G/E cyclic. There is an element $x \in G - E$ of order $2^{s+1}, s \geq 1$, such that $G = \langle E, x\rangle$, $E \cap \langle x\rangle = \langle n\rangle$, and*

$$t^x = tz, \quad z^x = zn^\epsilon, \quad \epsilon = 0, 1,$$

where in case $\epsilon = 1$ we must have $s > 1$, and we have in that case $G' = \langle n, z\rangle \cong E_4$ and $Z(G) = \langle x^4\rangle$. If $\epsilon = 0$, then G is a minimal nonabelian nonmetacyclic group. In any case, $\langle x^2\rangle$ is normal in G, $G/\langle x^2\rangle \cong D_8$, and $\Phi(G) = \langle x^2, z\rangle$ is abelian of type $(2^s, 2)$.

LEMMA 1.4: *Let V be a minimal non-quaternion-free 2-group. Then there is a normal subgroup U of V such that $V/U \cong Q_8$ and $U \leq \Phi(V)$ so that $d(V) = 2$. We have $\Phi(V)/U = Z(V/U)$ so that for each $x \in V - \Phi(V)$, $x^2 \in \Phi(V) - U$. In particular, there are no involutions in $V - \Phi(V)$.*

Proof: Trivial. ∎

We use very often the following result of A. Mann.

LEMMA 1.5 ([1, Lemma 64.1(u)]): *If A and B are two distinct maximal subgroups of a p-group G, then $|G' : (A'B')| \leq p$.*

For completeness we also state Iwasawa's result in a suitable form.

PROPOSITION 1.6 ([2]): *A 2-group G is modular if and only if G is D_8-free. A 2-group G is modular and Q_8-free if and only if G possesses a normal abelian subgroup A with cyclic G/A and there is an element $g \in G$ and an integer $s \geq 2$ such that $G = \langle A, g\rangle$ and $a^g = a^{1+2^s}$ for all $a \in A$ (and so if $\exp(G) \leq 4$, then G is abelian).*

THEOREM 1.7 (Main Theorem) (B. Wilkens): *A finite 2-group G is nonmodular and quaternion-free if and only if G is one of the following groups:*

(a) *G is a semidirect product $\langle x\rangle \cdot N$, where N is a maximal abelian normal subgroup of G with $\exp(N) > 2$ and, if t is the involution in $\langle x\rangle$, then every element in N is inverted by t.*
   (Wilkens group of type (a) with respect to N)
(b) *$G = N\langle x\rangle$, where N is a maximal elementary abelian normal subgroup of G and $\langle x\rangle$ is not normal in G.*
   (Wilkens group of type (b) with respect to N)

(c) $G = \langle N, x, t\rangle$, *where* N *is an elementary abelian normal subgroup of* G *and* t *is an involution with* $[N, t] = 1$. *If* $o(xN) = 2^k$, *then* $G/N \cong M_{2^{k+1}}$, $k \geq 3$, *and* $x^{2^k} \neq 1$; *furthermore,* $[x^{2^{k-1}}, N] = 1$ *and* $\langle t, x^{2^{k-1}}\rangle \cong D_8$. **(Wilkens group of type (c) with respect to** N, x, t**)**

We analyze now in great detail the above Wilkens groups of types (a), (b), and (c).

PROPOSITION 1.8: *Let* G *be a Wilkens group of type (a) with respect to* N. *Then* $\Omega_1(G) = N\langle t\rangle$, *where* t *is an involution in* $G - N$ *(acting invertingly on* N*) and* N *is a characteristic subgroup of* $\Omega_1(G)$. *The factor group* $G/\Omega_1(G)$ *is cyclic (since* G/N *is cyclic). If* G *is a Wilkens group of type (a) with respect to* N_1, *then* $N = N_1$. *Also,* G *is not* D_8*-free but* G *is* Q_8*-free. If* $z \in \Omega_1(Z(G))$, *then* $G/\langle z\rangle$ *is either abelian or a Wilkens group of type (a) or (b).*

Proof: Since G/N is cyclic, we have $\Omega_1(G) \leq N\langle t\rangle$. All elements in Nt are involutions and $\langle Nt\rangle = N\langle t\rangle$, and so $\Omega_1(G) = N\langle t\rangle$.

Suppose that N is not a characteristic subgroup of $\Omega_1(G)$. Then there is an automorphism α of $\Omega_1(G)$ such that $N^\alpha \neq N$ so that $\Omega_1(G) = NN^\alpha$ and $|\Omega_1(G) : (N \cap N^\alpha)| = 4$. Let t' be an involution in $N^\alpha - N$ so that t' inverts and centralizes each element in $N \cap N^\alpha$. But then $N \cap N^\alpha$ is elementary abelian and so also $N^\alpha = \langle N \cap N^\alpha, t'\rangle$ is elementary abelian, contrary to $\exp(N^\alpha) = \exp(N) > 2$.

Suppose that G is also a Wilkens group of type (a) with respect to N_1 which is distinct from N. Let t_1 be an involution in $G - N_1$ which inverts N_1. By the above, $\Omega_1(G) = N_1\langle t_1\rangle = N\langle t\rangle$. We have $|\Omega_1(G) : N_1| = 2$ and so $\Omega_1(G) = NN_1$. Let $t_0 \in N_1 - N$ so that the involution t_0 inverts and centralizes each element in $N \cap N_1$. But then $N \cap N_1$ is elementary abelian and $\exp(N_1) = 2$, a contradiction.

Since t inverts N and $\exp(N) > 2$, G is not D_8-free. Suppose that G is not Q_8-free. Let V be a minimal non-Q_8-free subgroup of G so that V has a normal subgroup U with $V/U \cong Q_8$ and $\Phi(V) > U$. Since $V \not\leq N$ and G/N is cyclic, we see that $V/(V \cap N)$ is nontrivial cyclic. Let t' be an element in $V - N$ such that $(t')^2 \in N$. Then Nt' is the involution in G/N and so all elements in Nt' are involutions. In particular, all elements in the set $S = (V \cap N)t'$ are involutions. By Lemma 1.4, $S \leq \Phi(V)$ and so also $\langle S\rangle = (V \cap N)\langle t'\rangle \leq \Phi(V)$. But then $V/\Phi(V)$ is cyclic, a contradiction.

Let z be an involution in $Z(G)$. We want to determine the structure of $G/\langle z\rangle$ and we know that G is a semidirect product of $\langle x\rangle$ and N, and let t be the involution in $\langle x\rangle$. We have $z \in N$ and t inverts each element in N and so t inverts each element in $N/\langle z\rangle$. If $\exp(N/\langle z\rangle) > 2$, we see that $G/\langle z\rangle$ is a Wilkens group of type (a).

Suppose that $\exp(N/\langle z\rangle) = 2$. Set $E = \langle t\rangle N$. Then $E/\langle z\rangle = \Omega_1(G/\langle z\rangle)$ is a maximal elementary abelian normal subgroup of $G/\langle z\rangle$. If $\langle x, z\rangle = \langle x\rangle \times \langle z\rangle$ is not normal in G, then $G/\langle z\rangle$ is a Wilkens group of type (b). Suppose that $\langle x, z\rangle$ is normal in G. Then $G' \leq \langle x, z\rangle \cap N = \langle z\rangle$, which implies that $G' = \langle z\rangle$. In that case $G/\langle z\rangle$ is abelian and we are done. ∎

PROPOSITION 1.9: *A 2-group G is a Wilkens group of type (b) with respect to E if and only if G possesses a maximal normal elementary abelian subgroup E such that G/E is cyclic and G is not D_8-free. Let G be a Wilkens group of type (b) with respect to E. Then G is Q_8-free. We have $|\Omega_1(G) : E| \leq 2$ and $G/\Omega_1(G)$ is cyclic (since G/E is cyclic). If $|\Omega_1(G) : E| = 2$, then G has exactly two maximal normal elementary abelian subgroups E and E_1 and we have $\Omega_1(G) = EE_1$. In that case, if G is a Wilkens group of type (b) with respect to E_1 (also), then $|\Omega_1(G) : E_1| = 2$ and $\Omega_1(G) \cong D_8 \times E_{2^s}$. Let $z \in \Omega_1(Z(G))$. Then $G/\langle z\rangle$ is either abelian or a Wilkens group of type (b) or $G/\langle z\rangle \cong D \times F$, where $\exp(F) \leq 2$ and either $D \cong D_8$ or $D \cong M_{2^n}, n \geq 4$ (in which case $G/\langle z\rangle$ is modular and nonabelian). Finally, if G is any 2-group with an elementary abelian normal subgroup E_0 such that $G = \langle E_0, y\rangle$ (and so G/E_0 is cyclic) and $\langle y\rangle$ is not normal in G, then G is a Wilkens group of type (b) (with respect to any maximal elementary abelian normal subgroup E of G containing E_0).*

Proof: Let G be a nonmodular 2-group possessing a maximal elementary abelian normal subgroup E such that G/E is cyclic. We have $G = \langle E, x\rangle$ for some $x \in G$ and, if $\langle x\rangle$ is not normal in G, then G is a Wilkens group of type (b). Assume that $\langle x\rangle$ is normal in G. In that case, $G' \leq \langle x\rangle \cap E$ and $|\langle x\rangle \cap E| \leq 2$. Since G is nonmodular (and so nonabelian), $G' = \langle x\rangle \cap E = \langle z\rangle \cong C_2$. We have $[x, E] = \langle z\rangle$ and so $|G : C_G(x)| = 2$. We set $E_1 = C_E(x)$ so that $|E : E_1| = 2$. Let t be an involution in $E - E_1$ and let V be a complement of $\langle z\rangle$ in E_1 so that $G = V \times \langle x, t\rangle$. If $|G/E| = 2^s > 2$, then $\langle x, t\rangle \cong M_{2^{s+2}}, s \geq 2$, and so for each $a \in A = \langle x\rangle \times V$, $a^t = a^{1+2^s}$. But Proposition 1.6 implies that G is modular, a contradiction. Hence $|G/E| = 2$ and $\langle x, t\rangle \cong D_8$. In that case $\tilde{x} = xt$ is an involution in $G - E$, $G = E\langle\tilde{x}\rangle$, $\langle\tilde{x}\rangle$ is not normal in G, and so G is a Wilkens group of type (b).

Conversely, let G be a Wilkens group of type (b) with respect to E so that $G = \langle E, g\rangle$, where E is a maximal normal elementary abelian subgroup of G

and $\langle g\rangle$ is not normal in G. Set $Z = \langle g\rangle \cap E$ so that $|Z| \le 2$ and $Z \le Z(G)$. Set $S = N_G(\langle g\rangle)$ so that $S \ne G$ and $S \cap E < E$. Since $N_G(S) = \langle g\rangle N_E(S)$, there is an involution $n \in E - S$ normalizing S. We have

$$[n,g] \in S \cap E \quad \text{and} \quad 1 \ne u = [n,g] \notin \langle g\rangle$$

since n does not normalize $\langle g\rangle$. We have

$$[n,g] = ng^{-1}ng = nn^g = u.$$

On the other hand, $\Phi(S) = \langle g^2\rangle$ and so $\langle g^2\rangle$ is normal in $\langle S, n\rangle$ so that $n^{g^2} = nz$ with $z \in Z$. Hence $\langle g\rangle$ normalizes $\langle n, n^g, Z\rangle$ and acts nontrivially on the four-group $\langle n, n^g, Z\rangle/Z$, where $\langle g^2\rangle \ge Z$. It follows that $\langle n, g\rangle/\langle g^2\rangle \cong D_8$ and so G is not D_8-free.

From now on we denote by G a Wilkens group of type (b) with respect to E. First of all, G is Q_8-free. Indeed, if V is a minimal non-Q_8-free subgroup of G, then (by Lemma 1.4) there are no involutions in $V - \Phi(V)$ so that $\Phi(V) \ge V \cap E$. But then $V/\Phi(V)$ is cyclic since $V/V \cap E$ is cyclic, a contradiction.

Set $W/E = \Omega_1(G/E)$ so that $|W : E| = 2$ and $\Omega_1(G) \le W$. It follows that $|\Omega_1(G) : E| \le 2$. Suppose that $|\Omega_1(G) : E| = 2$ so that $\Omega_1(G) = W$ and there is an involution $t \in W - E$. Since E is a maximal normal elementary abelian subgroup of G, $\langle t\rangle$ is not normal in W and so W is a Wilkens group of type (b) (with respect to E). Set $E_1 = C_W(t)$ so that $E_1 - E$ is the set of all involutions in $W - E$. Since $\langle E_1 - E\rangle = E_1$, E_1 is normal in G and E and E_1 are the only maximal normal elementary abelian subgroups of G. If G is a Wilkens group of type (b) also with respect to E_1, then G/E_1 must be cyclic and so $|W : E_1| = 2$ and in that case $W = \Omega_1(G) \cong D_8 \times E_{2^s}$. Indeed, in that case take $n \in E - E_1$ so that $\langle n, t\rangle \cong D_8$, $E \cap E_1 = Z(W)$, and if V is a complement of $\langle [n,t]\rangle$ in $E \cap E_1$, then $W = V \times \langle n, t\rangle$.

Let z be a central involution in G, where $G = \langle E, x\rangle$ and $\langle x\rangle$ is not normal in G. Then $z \in E$. If $z \in \langle x\rangle$, then the fact that $\langle x\rangle/\langle z\rangle$ is not normal in $G/\langle z\rangle$ gives that $G/\langle z\rangle$ is a Wilkens group of type (b). Suppose that $z \notin \langle x\rangle$. If $\langle x, z\rangle$ is not normal in G, then again $G/\langle z\rangle$ is a Wilkens group of type (b). Assume that $\langle x, z\rangle = \langle x\rangle \times \langle z\rangle$ is normal in G. If $\langle x\rangle \cap E = \{1\}$, then $\langle x, z\rangle \cap E = \langle z\rangle$ and $G' \le \langle x, z\rangle \cap E = \langle z\rangle$ and so $G/\langle z\rangle$ is abelian. Assume that $\langle x\rangle \cap E \ne \{1\}$ so that $\langle x, z\rangle \cap E = \Omega_1(\langle x\rangle) \times \langle z\rangle \cong E_4$ and suppose that $G/\langle z\rangle = \bar{G}$ is nonabelian. Then $\bar{G} = \langle \bar{x}\rangle \bar{E}$ with $\langle \bar{x}\rangle \cap \bar{E} \cong C_2$ and both $\langle \bar{x}\rangle$ and the elementary abelian group $\bar{E}$ are normal in $\bar{G}$ so that $\bar{G}' = \langle \bar{x}\rangle \cap \bar{E}$. Let $\bar{t}$ be an involution in $\bar{E}$ which does not centralize $\langle \bar{x}\rangle$. If $o(\bar{x}) = 4$, then $\langle \bar{x}, \bar{t}\rangle \cong D_8$, and if $o(\bar{x}) > 4$, then $\langle\bar{x},\bar{t}\rangle \cong M_{2^n}, n \geq 4$. We have

$$\bar{E} = (\langle\bar{x}\rangle \cap \bar{E}) \times \langle\bar{t}\rangle \times \bar{V},$$

where $(\langle\bar{x}\rangle \cap \bar{E}) \times \bar{V} = C_{\bar{E}}(\bar{x})$ so that $\bar{G} = \bar{V} \times \langle\bar{x},\bar{t}\rangle$ and we are done. ∎

PROPOSITION 1.10: *Let G be a Wilkens group of type (c) with respect to N, x, t. Then G is Q_8-free but is not D_8-free. We have $\Omega_1(G) = \langle x^{2^{k-1}}, t\rangle N \cong D_8 \times E_{2^s}$ and $G = \Omega_1(G)\langle x\rangle$ so that $G/\Omega_1(G)$ is cyclic of order ≥ 4. Also, N is the unique maximal normal elementary abelian subgroup of G. No subgroup X of order ≥ 8 in $\langle x\rangle$ is normal in G. The involution $z = x^{2^k}$ lies in $G' \cap Z(G)$ and $G/\langle z\rangle$ is a Wilkens group of type (b). If $z' \in \Omega_1(Z(G))$ and $z' \neq z$, then $z' \in N$ and $G/\langle z'\rangle$ is a Wilkens group of type (c).*

Proof: By definition, $\langle x^{2^{k-1}}, t\rangle \cong D_8$ and so G is not D_8-free. We set $a = x^{2^{k-1}}$ and $z = a^2$. We have $(G/N)' = \Omega_1(Z(G/N)) = (\langle a\rangle N)/N$ and $\Omega_1(G/N) = (\langle t, a\rangle N)/N = W/N$, where $W = \langle t, a\rangle N$ and $Z(W) = N$. Each involution in G must be contained in W and $W = \Omega_1(W)$ and so $\Omega_1(G) = W \cong D_8 \times E_{2^s}$ and $G = \Omega_1(G)\langle x\rangle$, so that $G/\Omega_1(G)$ is cyclic of order ≥ 4. By the structure of G/N, if X is a normal subgroup of G with $N \leq X \leq W$, then $X \in \{N, W, \langle a\rangle N\}$, where $\langle a\rangle N$ is abelian of type $(4, 2, \ldots, 2)$. This gives that N is a maximal normal elementary abelian subgroup of G. Let N_1 be any maximal normal elementary abelian subgroup of G. Then $N_1 \leq W$ and assume that $N_1 \neq N$. Since N_1 does not cover W/N (since all elements in $(\langle a\rangle N) - N$ are of order 4), we get $|(NN_1) : N| = 2$, NN_1 is normal in G and so (by the above) $NN_1 = \langle a\rangle N$, a contradiction. We have proved that N is the unique maximal normal elementary abelian subgroup of G.

Let $Y \leq \langle x\rangle$ with $|Y| \geq 8$ and assume that t normalizes Y. Since t inverts $\langle a\rangle$ and $\langle a\rangle < Y$, it follows that $Y\langle t\rangle$ is of maximal class and order $2^m, m \geq 4$, and so $(Y\langle t\rangle)/\langle z\rangle \cong D_{2^{m-1}}$ is isomorphic to a proper subgroup of G/N. But $G/N \cong M_{2^n}, n \geq 4$, is minimal nonabelian, a contradiction. We have proved that Y is not normal in G.

Suppose that G is not Q_8-free and let V be a minimal non-Q_8-free subgroup of G. By Lemma 1.4, there are no involutions in $V - \Phi(V)$ and so $\Phi(V) \geq V \cap N$. On the other hand, $\Phi(V)$ is contained in the maximal subgroup $\langle x\rangle N$ of G, and note that $\Omega_1(\langle x\rangle N) = N$ (since $N \cap \langle x\rangle = \langle z\rangle$ and a centralizes N). It follows that $\Omega_1(\Phi(V)) = V \cap N$. Note that G/N has exactly three involutions: Na, Nt, and $N(at)$, where all elements in the coset Na are of order 4 and all elements in cosets Nt and $N(at)$ are involutions. Let $(V \cap N)s$ $(s \in V)$ be an involution in

$V/(V\cap N)$. Then Ns is an involution in G/N. If all elements in the coset Ns are involutions, then $s \in \Phi(V)$, contrary to the above fact that $\Omega_1(\Phi(V)) = V \cap N$. It follows that $Ns = Na$ and so $(V \cap N)s = (Na) \cap V$ is the unique involution in $V/(V\cap N)$. Since G/N is Q_8-free, we get that $V/(V\cap N)$ is cyclic. But then $V/\Phi(V)$ is also cyclic, a contradiction.

We shall determine the structure of $G/\langle z\rangle = \bar{G}$. Since $\overline{\Omega_1(G)} = \Omega_1(G)/\langle z\rangle$ is an elementary abelian normal subgroup of $\bar{G}$, $\bar{G} = \overline{\Omega_1(G)}\langle \bar{x}\rangle$, and $\langle \bar{x}\rangle$ is not normal in $\bar{G}$ (noting that $z \in \langle x\rangle$ and $\langle x\rangle$ is not normal in G), $\bar{G}$ is a Wilkens group of type (b). Let z' be an involution in $Z(G)$ and $z' \neq z$. Then $z' \in N$ and $\langle z'\rangle \cap \langle a,t\rangle = \{1\}$ so that $G/\langle z'\rangle$ is not D_8-free. Obviously, $G/\langle z'\rangle$ is a Wilkens group of type (c). ∎

Proposition 1.11: *Let G be one of the Wilkens groups. Suppose that there is an involution $z \in Z(G)$ such that $G/\langle z\rangle$ is modular (i.e., D_8-free).*

(i) *If $G/\langle z\rangle$ is abelian, then G is a Wilkens group of type (b) and, more precisely, $G = D \times E$, where $\exp(E) \leq 2$, and*

$$D = \langle x,t \mid x^{2^n} = t^2 = 1, n \geq 1, [x,t] = z, z^2 = [x,z] = [t,z] = 1\rangle.$$

(If $n = 1$, then $D \cong D_8$, and if $n > 1$, then D is minimal nonabelian nonmetacyclic with $\Omega_1(D) \not\leq Z(D)$.)

(ii) *If $G/\langle z\rangle$ is nonabelian, then there is another involution $z' \in Z(G)$ such that $\langle z'\rangle$ is a characteristic subgroup of G and $G/\langle z'\rangle$ is a Wilkens group of type (b) (and so nonmodular).*

Proof: (i) Suppose that $G/\langle z\rangle$ is abelian. Then $G' = \langle z\rangle$ and, by Propositions 1.8, 1.9, and 1.10, G is a Wilkens group of type (a) or (b).

Suppose that G is of type (a). Then $G = \langle x\rangle \cdot N$ (a semidirect product), where N is a maximal normal abelian subgroup of G with $\exp(N) > 2$, and if t is the involution in $\langle x\rangle$, then t inverts each element of N. Suppose $n \in N$ with $o(n) > 2$. Then $\langle n,t\rangle$ is dihedral and so $n^2 \in \langle n,t\rangle'$. It follows that $n^2 \in \langle z\rangle$ and therefore $N/\langle z\rangle$ is elementary abelian with $\mho_1(N) = \langle z\rangle$ and $|N : \Omega_1(N)| = 2$. Suppose $\langle x\rangle > \langle t\rangle$ and let $v \in \langle x\rangle$ with $v^2 = t$. Let $n \in N$ with $o(n) = 4$. We have $[n,v] \neq 1$ and so $[n,v] = z$, which gives $n^v = nz$. But then

$$n^t = n^{v^2} = (nz)^v = n^v z^v = nzz = n,$$

since $z \in Z(G)$. This is a contradiction and so $\langle x\rangle = \langle t\rangle$. If $n \in N$ with $o(n) = 4$, then $D = \langle n,t\rangle \cong D_8$. Let E be a complement of $\langle z\rangle$ in $\Omega_1(N)$, where t centralizes $\Omega_1(N)$. We get $G = \langle t \rangle \cdot N = D \times E$, where $D \cong D_8$ and $\exp(E) \le 2$.

Suppose that G is a Wilkens group of type (b). Then $G = \langle N, x \rangle$, where N is a maximal normal elementary abelian subgroup of G and $\langle x \rangle$ is not normal in G. We have $z \in N$ and $G' = \langle z \rangle$. If $z \in \langle x \rangle \cap N$, then $\langle x \rangle$ is normal in G, a contradiction. Hence $z \notin \langle x \rangle$ and $\langle x, z \rangle = \langle x \rangle \times \langle z \rangle$ is normal in G. Since $\langle x, z \rangle$ contains exactly two cyclic subgroups $\langle x \rangle$ and $\langle xz \rangle$ of index 2 not containing $\langle z \rangle$, we have $|G : N_G(\langle x \rangle)| = 2$. There is $t \in N - N_G(\langle x \rangle)$ such that $x^t = xz$. Also, note that $N_G(\langle x \rangle)$ centralizes $\langle x \rangle$ (since $G' = \langle z \rangle$). Let E be a complement of $\langle z, \langle x \rangle \cap N \rangle$ in $N_N(\langle x \rangle)$. Then $G = D \times E$, where

$$D = \langle x, t | x^{2^n} = t^2 = 1, n \ge 1, [x, t] = z, z^2 = [x, z] = [t, z] = 1 \rangle.$$

(ii) Suppose that $G/\langle z \rangle$ is nonabelian. By Propositions 1.8, 1.9, and 1.10, G is a Wilkens group of type (b). Then $G = \langle N, x \rangle$, where N is a maximal normal elementary abelian subgroup of G and $\langle x \rangle$ is not normal in G. If $z \in \langle x \rangle \cap N$, then the fact that $\langle x \rangle$ is not normal in G gives that $\langle x \rangle / \langle z \rangle$ is not normal in $G/\langle z \rangle$ and so $G/\langle z \rangle$ is a Wilkens group of type (b), a contradiction. Hence $z \notin \langle x \rangle$. If $\langle x, z \rangle$ is not normal in G, then again $G/\langle z \rangle$ is a Wilkens group of type (b), a contradiction. Hence $\langle x, z \rangle = \langle x \rangle \times \langle z \rangle$ is normal in G. Assume first that $\langle x \rangle \cap N = \{1\}$. Then $\langle x, z \rangle \cap N = \langle z \rangle$ and $G' \le \langle x, z \rangle \cap N = \langle z \rangle$ and so $G/\langle z \rangle$ is abelian, a contradiction. Hence $\langle x \rangle \cap N > \{1\}$ and so $\langle x, z \rangle \cap N = \Omega_1(\langle x \rangle) \times \langle z \rangle$. Since $G' \le \langle x, z \rangle \cap N$ and $G/\langle z \rangle$ is nonabelian (by assumption), we get $1 \ne |G'| \le 4$ and $G' \not\le \langle z \rangle$. We have $\Omega_1(\langle x \rangle) \le Z(G)$. Set $\Omega_1(\langle x \rangle) = \langle x_0 \rangle$ and $S = N_G(\langle x \rangle)$ so that $\langle x_0, z \rangle \le Z(G)$, $|G : S| = 2$ and $|N : S \cap N| = 2$, because $\langle x \rangle$ is not normal in G and the abelian normal subgroup $\langle x, z \rangle$ has exactly two cyclic subgroups $\langle x \rangle$ and $\langle xz \rangle$ of index 2. Therefore, we have $[x, s] = z$ or $[x, s] = x_0 z$ for an $s \in N - S$ and so

$$\Phi(G) = \langle x^2, [\langle x \rangle, N] \rangle = \langle x^2 \rangle \times \langle z \rangle.$$

Suppose o(x)≥ 8 so that $\Phi(\Phi(G)) = \langle x^4 \rangle \ge \langle x_0 \rangle$ and $\langle x_0 \rangle$ is a characteristic subgroup of G. But $\langle x \rangle / \langle x_0 \rangle$ is not normal in $G/\langle x_0 \rangle$ (since $\langle x \rangle$ is not normal in G) and so $G/\langle x_0 \rangle$ is a Wilkens group of type (b) and we are done in this case.

Suppose $o(x) = 4$ so that $x^2 = x_0$. If $\langle x \rangle$ is not central in S, then there is an involution t in $S \cap N$ which inverts $\langle x \rangle$ and so $\langle x, t \rangle \cong D_8$. But then $G/\langle z \rangle$ is not D_8-free, contrary to our assumption that $G/\langle z \rangle$ is modular. Hence $\langle x \rangle$ is central in S and so $G' = \langle [x, s] \rangle = \langle x_0 z \rangle$ (since in case $G' = \langle [x, s] \rangle = \langle z \rangle$, $G/\langle z \rangle$ would be abelian). But then $\langle x, s \rangle$ is the minimal nonabelian nonmetacyclic group of order 2^4 with $\langle x, s\rangle' = \langle x_0 z\rangle$ and $\langle x, s\rangle/\langle z\rangle \cong D_8$, contrary to our assumption that $G/\langle z\rangle$ is modular. ∎

LEMMA 1.12: *Let G be a Wilkens group of type (b) with respect to N. Suppose in addition that $\Omega_1(G) = N$. Then for each element $g \in G$ such that $G = \langle N, g\rangle$, $N \cap \langle g\rangle = \langle g_0\rangle$ is of order 2 and $G/\langle g_0\rangle$ is also a Wilkens group of type (b) (and so $G/\langle g_0\rangle$ is nonmodular).*

Proof: It is enough to show that $\langle g\rangle$ is not normal in G (because then $\langle g\rangle/\langle g_0\rangle$ is also not normal in $G/\langle g_0\rangle$). Suppose false. Then $G' \le N \cap \langle g\rangle = \langle g_0\rangle$ and so $G' = \langle g_0\rangle$. We have $|G : C_G(g)| = 2$ and let t be an involution in $N - C_N(g)$. If $o(g) = 4$, then $\langle g, t\rangle \cong D_8$ and so gt is an involution in $G - N$, a contradiction. Hence $o(g) > 4$ and $\langle g, t\rangle \cong M_{2^n}, n \ge 4$. If V is a complement of $\langle g_0\rangle$ in $C_N(g)$, then $G = V \times \langle g, t\rangle$. But then G is modular (see Proposition 1.6), a contradiction. ∎

LEMMA 1.13 ([3, Proposition 1.10]): *Let τ be an involutory automorphism acting on an abelian group B so that $C_B(\tau) = W_0$ is contained in $\Omega_1(B)$. Then τ acts invertingly on $\mho_1(B)$ and on B/W_0.*

All the above results will be used freely in the proof of the Main Theorem. The reader should be acquainted with the structure of minimal nonabelian p-groups (see [1, Lemma 65.1]). Also, we use often the relation $|G| = p|G'||Z(G)|$, where G is a nonabelian p-group possessing an abelian maximal subgroup (see [1, Lemma 1.1]).

2. Proof of the Main Theorem

Let G be a nonmodular quaternion-free 2-group of a smallest possible order which is not isomorphic to any Wilkens group. Hence any proper nonmodular subgroup and any proper nonmodular factor group is isomorphic to a Wilkens group. We shall study such a minimal counter-example G and our purpose is to show that such a group G does not exist.

(i) There is a central involution z of G such that $G/\langle z\rangle$ is nonmodular (and so $G/\langle z\rangle$ is isomorphic to a Wilkens group).

Suppose false. Then for each $z \in \Omega_1(Z(G))$, $G/\langle z\rangle$ is modular. Let $z_0 \in \Omega_1(Z(G))$ so that $G/\langle z_0\rangle$ is modular. Since G is nonmodular, there is a minimal nonmodular subgroup K of G which is isomorphic to a group of Lemma 1.3. Obviously, K is a Wilkens group of type (b) and so $K \ne G$. Since $G/\langle z_0\rangle$ is modular, we have $z_0 \in K$. If $K \cong D_8$, then $\langle z_0 \rangle = \Omega_1(Z(G)) = K'$. Suppose that K has a normal elementary abelian subgroup $E = \langle n, z, t \rangle$ of order 8 such that $K = \langle E, x \rangle$, $o(x) = 2^{s+1}, s \geq 1$, $E \cap \langle x \rangle = \langle n \rangle$, $t^x = tz$, $z^x = zn^\epsilon, \epsilon = 0, 1$, $\Omega_1(K) = E$, and in case $\epsilon = 1$ we have $s > 1$ and $Z(K) = \langle x^4 \rangle$. If $\epsilon = 1$, then we must have $z_0 = n$. But then $K/\langle z_0 \rangle$ is nonmodular since $K/\langle x^2 \rangle \cong D_8$. Hence we have $\epsilon = 0$, in which case K is minimal nonabelian nonmetacyclic with $Z(K) = \langle x^2, z \rangle = \Phi(K)$ and $K' = \langle z \rangle$. Since $K/\langle z_0 \rangle$ is modular (and $K/\langle n \rangle$ is nonmodular), we have either $z_0 = z$ (and then $K/\langle z_0 \rangle$ is abelian) or $z_0 = zn$ (in which case $s > 1$ and $K/\langle zn \rangle \cong M_{2^{s+2}}$).

Let H be a maximal subgroup of G containing K. Since H is nonmodular and $H \neq G$, H is a Wilkens group. Since $H/\langle z_0 \rangle$ is modular, we may use Proposition 1.11. If $H/\langle z_0 \rangle$ is nonabelian, then there is another involution z_0' in $Z(H)$ such that $\langle z_0' \rangle$ is a characteristic subgroup in H and $H/\langle z_0' \rangle$ is nonmodular. But then $z_0' \in Z(G)$ and $G/\langle z_0' \rangle$ is nonmodular, contrary to our assumption. Hence $H/\langle z_0 \rangle$ must be abelian and so $K/\langle z_0 \rangle$ is also abelian. In particular, $z_0 = z$, where $\langle z \rangle = K'$. In any case $\Omega_1(Z(G)) = \langle z \rangle$ is of order 2. By Proposition 1.11(a), we have $H = D \times E_0$, where $\exp(E_0) \leq 2$ and

$$D = \langle y, t | y^{2^m} = t^2 = 1, m \geq 1, [y, t] = z, z^2 = [z, y] = [z, t] = 1 \rangle.$$

If $m = 1$, then $D \cong D_8$, and if $m > 1$, then D is minimal nonabelian nonmetacyclic with $E_8 \cong \Omega_1(D) \not\leq Z(D)$ and $z = z_0$, where $\langle z \rangle = D' = H' = \Omega_1(Z(G))$.

Suppose $m = 1$. Then $Z(H) = \langle z \rangle \times E_0$ is elementary abelian. If $|E_0| \geq 4$, then acting with an element $x \in G - H$ on $Z(H)$, we see that $|C_{Z(H)}(x)| \geq 4$ and $C_{Z(H)}(x) \leq Z(G)$, contrary to the fact that $\Omega_1(Z(G))$ is of order 2. Hence $|E_0| \leq 2$. If $D = H \cong D_8$, then $C_G(D) \leq D$ would imply that G is of maximal class and then $G/\langle z \rangle \cong D_8$, a contradiction. If $D = H \cong D_8$ and $C_G(D) \not\leq D$, then Lemma 1.1 implies that $G \cong D_8 \times C_2$, contrary to $|\Omega_1(Z(G))| = 2$. Hence we must have $H = D \times \langle t \rangle$, where t is an involution with $C_G(t) = H$. Since $H/\langle z \rangle$ is elementary abelian, $\exp(G/\langle z \rangle) \leq 4$ and therefore $G/\langle z \rangle$ is abelian since $G/\langle z \rangle$ is modular (and Q_8-free) of exponent ≤ 4. In particular, D is normal in G. We have $C_H(D) = \langle z, t \rangle$. If $C_G(D) > \langle z, t \rangle$, then $C_G(t) = G$, a contradiction. Hence $C_G(D) = \langle z, t \rangle$ and $\mathrm{Aut}(D_8) \cong D_8$ implies that $G/\langle z, t \rangle \cong D_8$. This contradicts our assumption that $G/\langle z \rangle$ is modular.

Suppose $m > 1$. Here $Z(D) = \Phi(D) = \langle y^2, z \rangle$ is abelian of type $(2^{m-1}, 2)$ and $Z(H) = \langle y^2, z \rangle \times E_0$ so that $\mho_1(Z(H)) = \langle y^4 \rangle$. If $m > 2$, then $\Omega_1(\langle y^4 \rangle)$ is of order 2 and $\Omega_1(\langle y^4 \rangle) \leq Z(G)$, contrary to the fact that $\Omega_1(Z(G)) = \langle z \rangle$. Thus we have $m = 2$, $|D| = 2^4$, and $Z(H) = \langle y^2, z \rangle \times E_0$ is elementary abelian. Suppose that $E_0 \neq \{1\}$. Then acting with an element $x \in G - H$ on $Z(H)$, we get $|C_{Z(H)}(x)| \geq 4$ and $C_{Z(H)}(x) \leq Z(G)$, a contradiction. It follows $D = H$ and so $|G| = 2^5$.

If $x \in G - H$ is of order 8, then $x^4 \in Z(H) - \langle z \rangle$ (with $\langle z \rangle = H'$) since $\Phi(H) = Z(H)$ and z is not a square in H. But then $Z(H) \leq Z(G)$, a contradiction. Hence $\exp(G) = 4$ and the fact that $G/\langle z \rangle$ is modular gives that $G/\langle z \rangle$ is abelian. It follows that $G' = H' = \langle z \rangle$. Since $H = D = \langle y, t \rangle$ and $|G : C_G(y)| = |G : C_G(t)| = 2$, we get $|G : C_G(H)| \leq 4$. But $|H : C_H(H)| = |H : Z(H)| = 4$ and so $C_G(H)$ must cover G/H. But then $E_4 \cong Z(H) \leq Z(G)$, a final contradiction.

(ii) The factor group $G/\langle z \rangle$ ($z \in \Omega_1(Z(G))$) is not isomorphic to a Wilkens group of type (a).

Suppose false. Then $G/\langle z \rangle$ has a maximal normal abelian subgroup $N/\langle z \rangle$ of exponent > 2 such that G/N is cyclic of order ≥ 2 and, if $L/N = \Omega_1(G/N)$, then for each element $x \in L - N$, $x^2 \in \langle z \rangle$, and x inverts each element of $N/\langle z \rangle$.

If all elements in $L - N$ are involutions, then each $y \in L - N$ inverts each element in N, which implies that N is abelian of exponent > 2, N is a maximal normal abelian subgroup of G and, if $G = \langle N, g \rangle$, then G is a semidirect product of N and $\langle g \rangle$ and the involution in $\langle g \rangle$ acts invertingly on N. Thus, G is a Wilkens group of type (a), a contradiction. Hence, there is $v \in L - N$ with $v^2 = z$.

Let n be an element of order 8 in N. Then $n^v = n^{-1}z^\epsilon$ ($\epsilon = 0, 1$), and therefore $(n^2)^v = (n^{-1}z^\epsilon)^2 = n^{-2}$, contrary to Lemma 1.1. We have proved that

$$\exp(N) = \exp(N/\langle z \rangle) = 4.$$

Suppose that N is abelian. Let s and l be elements of order 4 in N such that $\langle s \rangle \cap \langle l \rangle = \{1\}$. In that case $o(sl) = 4$ and, using Lemma 1.1, we get $s^v = s^{-1}z$, $l^v = l^{-1}z$ and consequently $(sl)^v = s^{-1}zl^{-1}z = (sl)^{-1}$, a contradiction. Hence N is abelian of type $(4, 2, \ldots, 2)$ and, since $\exp(N/\langle z \rangle) = 4$, $z \notin \mho_1(N)$. We set $\mho_1(N) = \langle t \rangle$ with $t \neq z$ and so $|N : \Omega_1(N)| = 2$, where each element in $N - \Omega_1(N)$ is of order 4 and has square equal to t. Also, $\langle t \rangle = \mho_1(N)$ is central in G. Let $a \in N - \Omega_1(N)$ so that $a^2 = t$ and (by Lemma 1.1) $a^v = a^{-1}z = a(zt)$. By Lemma 1.2, $\langle v, a \rangle$ is minimal nonabelian nonmetacyclic of order 2^4 with $[v, a] = zt$ and va is an involution. Suppose that v does not commute with an involution $u \in \Omega_1(N)$. Then $u^v = uz$, $o(au) = 4$, and $(au)^v = a^{-1}zuz = (au)^{-1}$, contrary to Lemma 1.1. It follows that $C_N(v) = \Omega_1(N)$, $[\langle v \rangle, N] = \langle tz \rangle$, $L' = \langle tz \rangle$, $\Omega_1(N) = Z(L)$, $\Phi(L) = \langle z, t \rangle$, where $\langle t \rangle = \mho_1(N)$. We have

$$L - N = v\Omega_1(N) \cup (va)\Omega_1(N),$$

where all elements in $v\Omega_1(N)$ are of order 4 and their squares are equal z and all elements in $(va)\Omega_1(N)$ are involutions. Thus $E = \langle va\rangle\Omega_1(N) = \Omega_1(L)$ is elementary abelian of index 2 in L and also $E = \Omega_1(G)$ (since G/N is cyclic) is the unique maximal normal elementary abelian subgroup of G. Now, L/E is a normal subgroup of order 2 in G/E with cyclic factor group G/L. Thus G/E is abelian. If G/E were cyclic, then the fact that G is not D_8-free gives that G is a Wilkens group of type (b), a contradiction. Thus G/E is abelian of type $(2^s, 2), s \geq 1$. If $s > 1$, then we set $K/E = \Omega_1(G/E) \cong E_4$. Since $K \geq L$, K is not D_8-free and so $K < G$ implies that K must be a Wilkens group. But $E = \Omega_1(K)$ and K/E is noncyclic, a contradiction (see Propositions 1.8 to 1.10). Hence $s = 1$, $G/E \cong E_4$, and so $\exp(G) = 4$. We have $G/N \cong C_4$ and so for each $x \in G - L$, $x^2 \in L - N$ and so $x^2 \in E - N$. Since the square of each element in G is contained in $\langle z\rangle$ or $\langle t\rangle$ or in $E - N$, it follows that tz is not a square in G. Hence $\Omega_1(G/\langle tz\rangle) = E/\langle tz\rangle$. If $G/\langle tz\rangle$ were nonmodular, then $G/\langle tz\rangle$ must be a Wilkens group and then $(G/\langle tz\rangle)/(E/\langle tz\rangle) \cong G/E$ must be cyclic, a contradiction. It follows that $G/\langle tz\rangle$ is modular and, since $\exp(G) = 4$, $G/\langle tz\rangle$ is abelian and so $G' = \langle tz\rangle$. Let $x \in G - L$ so that $x^2 = E - N$. Then $[v, x] \in \langle tz\rangle$ and so $[v, x^2] = [v, x]^2 = 1$. But then v centralizes E and, since $L = \langle E, v\rangle$, we get that L is abelian, a contradiction. We have proved that N is nonabelian and so $N' = \langle z\rangle$.

The subgroup N is nonmodular because a Q_8-free modular 2-group of exponent 4 is abelian. Let S be a minimal nonabelian subgroup of N. Then $S' = \langle z\rangle$ and S is normal in N. Since v acts invertingly on $N/\langle z\rangle$, v normalizes S and so S is normal in $L = \langle N, v\rangle$. Since exp $(S) = 4$ and S is Q_8-free, it follows that either $S \cong D_8$ or S is minimal nonabelian nonmetacyclic of order 2^4. If $S \cong D_8$, then $\mho_1(S) = \langle z\rangle$ and $N' = \langle z\rangle$ implies that $C_N(S)$ covers N/S. In that case, Lemma 1.1 implies that $C_N(S)$ is elementary abelian and so $\mho_1(N) = \langle z\rangle$, contrary to $\exp(N/\langle z\rangle) > 2$. It follows that we have the second possibility:

$$S = \langle a, t | a^4 = t^2 = 1, [a, t] = z, z^2 = [a, z] = [t, z] = 1\rangle.$$

We put $b = at$ and compute $b^2 = a^2t^2[t, a] = a^2z$, so that $S' = \langle z\rangle$, $o(b) = 4$, $\langle a\rangle \cap \langle b\rangle = \{1\}$, and $S = \langle a, b\rangle$ with $[a, b] = a^2b^2 = z$. Again, since $N' = S' = \langle z\rangle$, we get that $C_N(S)$ covers N/S and $C_N(S) \cap S = Z(S) = \Phi(S) = \langle z, a^2\rangle$. Suppose that $y \in C_N(S)$ is of order 4. Since $\langle a\rangle \cap \langle b\rangle = \{1\}$, there is an $s \in \{a, b\}$ so that $\langle s\rangle \cap \langle y\rangle = \{1\}$ and $o(sy) = 4$. We compute

$$(sy)^v = s^{-1}zy^{-1}z = s^{-1}y^{-1} = (sy)^{-1},$$

contrary to Lemma 1.1. We have proved that $C_N(S)$ is elementary abelian and

$N = SC_N(S)$, $S \cap C_N(S) = Z(S)$ and so the structure of N is completely determined.

We act with $\langle v\rangle$ on $S = \langle a, b\rangle$ and get (using Lemma 1.1) $a^v = a^{-1}z = a(a^2z)$, $b^v = b^{-1}z = ba^2$, which together with $v^2 = z$ determines uniquely the structure of $T = S\langle v\rangle$.

Suppose that v does not centralize $C_N(S) = Z(N)$. Then there is an involution $s \in C_N(S) - S$ such that $s^v = sz$. Then $o(as) = 4$ and $(as)^v = a^{-1}zsz = (as)^{-1}$, contrary to Lemma 1.1. Hence $C_N(S) = C_N(T)$ and $L = TC_L(T)$ with $T \cap C_L(T) = Z(T) = \langle z, a^2\rangle$.

We have $\Omega_1(S) = \langle z, a^2, ab\rangle \cong E_8$, $\langle z, a^2\rangle = Z(T)$,

$$(av)^2 = avav = av^2v^{-1}av = aza^{-1}z = 1,$$

and

$$(ab)^{av} = (abz)^v = a^{-1}zb^{-1}zz = a^{-1}b^{-1}z = aa^2bb^2z = aba^2(a^2z)z = ab.$$

Hence $F = \langle z, a^2, ab, av\rangle \cong E_{2^4}$ is an elementary abelian maximal subgroup of T and so, from $T' \geq \langle z, a^2\rangle$ and $|T| = 2^5 = 2|T'||Z(T)|$, it follows that $T' = Z(T) = \langle z, a^2\rangle$. Finally, $T/\langle z, a^2\rangle$ is elementary abelian and therefore $Z(T) = T' = \Phi(T) \cong E_4$, and so T is a special group of order 2^5. For each $x \in T - F$, $C_F(x) = Z(T)$, and so the set $T_0 = T - F$ contains exactly four square roots of z, four square roots of a^2, four square roots of za^2, and so T_0 must contain exactly four involutions in $T - S$. If t_0 is one of them, then $C_F(t_0) = Z(T)$ and F and $\langle z, a^2, t_0\rangle \cong E_8$ are the only maximal normal elementary abelian subgroups of T (containing all involutions of T).

We have $C_N(S) = C_N(T) = Z(L)$, and so if we set $U = FZ(L)$, $V = \langle z, a^2, t_0\rangle Z(L)$, then $L = UV$, $U \cap V = Z(L)$, U and V are the only maximal normal elementary abelian subgroup of L and they are of distinct orders, and so U and V are normal in G and $|U : Z(L)| = 4$, $|V : Z(L)| = 2$, $L' = \Phi(L) = \langle z, a^2\rangle$. For each $t_0 \in V - Z(L)$, $C_U(t_0) = Z(L)$, and so both U and V are self-centralizing in L. Also, U is the unique abelian maximal subgroup of L (otherwise, by a result of A. Mann, $|L'| \leq 2$). Now, G/N is cyclic, $L/N = \Omega_1(G/N)$, and so $\Omega_1(G) = L$, which is a Wilkens group of type (b) with respect to U. In particular, $L < G$.

If G/U is cyclic, then (since G is not D_8-free) G is a Wilkens group of type (b), a contradiction. Hence G/U is noncyclic. But $L/U \leq Z(G/U)$ and G/L is cyclic (since $L \geq N$ and G/N is cyclic) and so G/U is abelian of type $(2^n, 2)$. Set $K/U = \Omega_1(G/U) \cong E_4$. If $G \neq K$, then K is a Wilkens group with $\Omega_1(K) = L$.

By the structure of L, L does not have an abelian maximal subgroup of exponent > 2 and so K is not a Wilkens group of type (a). Also, $|K : \Omega_1(K)| = 2$ and so K is not a Wilkens group of type (c). Hence K must be a Wilkens group of type (b) with respect to U. But $K/U \cong E_4$, a contradiction. Hence $G = K$ and so $G/U \cong E_4$ implies $\exp(G) = 4$. Since $\Omega_1(G) = L$, all elements in $G - L$ are of order 4, and if $x \in G - L$, then $x^2 \in U - N$, where $N = \langle Z(L), ab, a\rangle$ and $U \cap N = Z(L) \times \langle ab\rangle$.

If $C_G(V) > V$, then $C_L(V) = V$ implies that there is $y \in G - L$ with $y^2 \in V$, contrary to $y^2 \in U - N$. We have proved that $C_G(V) = V$ and so G/V acts faithfully on V. We have $G/V \cong (U\langle x\rangle)/Z(L)$, where $U/Z(L) \cong E_4$ and $\langle x\rangle Z(L)/Z(L) \cong C_4$ with $x \in G - L$. Thus $G/V \cong D_8$ or $G/V \cong C_4 \times C_2$. But $L/V \cong U/Z(L) \cong E_4$ is a four-subgroup in G/V and, for each $x \in G - L$, $x^2 \in U - N$ so that $\langle x\rangle \cap V = \{1\}$. Hence all elements in $(G/V) - (L/V)$ are of order 4 and so $G/V \cong C_4 \times C_2$.

If $L' = \langle z, a^2\rangle = Z(L)$, then $V \cong E_8$. But $G/V \cong C_4 \times C_2$ cannot act faithfully on $V \cong E_8$. We have proved that $Z(L) > L'$ and so $|Z(L)| \geq 8$.

We act with $\langle x\rangle$ on $Z(L)$, where $x \in G - L$. Since $x^2 \in L$, $\langle x\rangle$ induces an automorphism of order ≤ 2 on $Z(L)$. Since $|Z(L)| \geq 8$, it follows that $|C_{Z(L)}(x)| \geq 4$. Suppose that x centralizes an involution $u \in Z(L) - L'$ so that $u \in Z(G)$. Since $G/\langle u\rangle$ is nonabelian and of exponent 4, $G/\langle u\rangle$ must be nonmodular. Thus $G/\langle u\rangle$ is a Wilkens group with $\Omega_1(G/\langle u\rangle) = L/\langle u\rangle$ because $\Omega_1(G) = L$ and u is not a square in G. Then $U/\langle u\rangle$ and $V/\langle u\rangle$ are the only maximal normal elementary abelian subgroups of $G/\langle u\rangle$ and both G/U and G/V are noncyclic and so $G/\langle u\rangle$ cannot be a Wilkens group of type (b). Also, $G/\langle u\rangle$ cannot be of type (c) since $\Omega_1(G/\langle u\rangle) = L/\langle u\rangle$ and $|G/L| = 2$. If $G/\langle u\rangle$ is a Wilkens group of type (a), then (by the first part of the proof of (ii)) L must possess a nonabelian maximal subgroup N_0 containing $\langle u\rangle$ such that $N_0' = \langle u\rangle$. This is a contradiction since $u \in Z(L) - L'$. We have proved that for each $x \in G - L$, $C_{Z(L)}(x) \leq L'$. Since $|C_{Z(L)}(x)| \geq 4$, we must have $C_{Z(L)}(x) = L' = Z(G) = \langle z, a^2\rangle$.

Let $x \in G - L$ so that $x^2 \in U - N$. We have $U = \langle z, a^2, ab, av\rangle Z(L)$ and so $(ab)^{x^2} = ab$. On the other hand, $x^2 \in L - N$ and x^2 acts invertingly on $N/\langle z\rangle$. Thus

$$a^{x^2} = a^{-1}z^{\epsilon}, \quad b^{x^2} = b^{-1}z^{\eta} \quad (\epsilon, \eta = 0, 1),$$

and so noting that $b^2 = a^2 z$ we get

$$ab = (ab)^{x^2} = a^{-1}z^{\epsilon}b^{-1}z^{\eta} = a^{-1}b^{-1}z^{\epsilon+\eta} = aa^2bb^2z^{\epsilon+\eta} = abz^{1+\epsilon+\eta},$$

which gives $\epsilon + \eta \equiv 1(\mathrm{mod}\, 2)$ and so $\epsilon = 1$ or $\eta = 1$.

Suppose $\epsilon = 1$. Then $a^{x^2} = a^{-1}z = a(a^2z)$ and we apply Lemma 1.2 in the group $G/\langle a^2z\rangle = \bar{G}$. We have (using the bar convention) $o(\bar{x}) = o(\bar{a}) = 4$ and $[\bar{x}^2, \bar{a}] = 1 = [\bar{x}, \bar{a}^2]$. If $[\bar{x}, \bar{a}] = 1$, then $[x, a] \in \langle a^2z\rangle$ (and $\langle x, a\rangle$ is of class ≤ 2 since $\langle x, a\rangle' \leq \langle a^2z\rangle$) and so $[x^2, a] = [x, a]^2 = 1$, a contradiction. Thus $[\bar{x}, \bar{a}] \neq 1$ and so (by Lemma 1.2) $o(\overline{xa}) = 2$. Hence $(xa)^2 \in \langle a^2z\rangle$, contrary to the fact that $xa \in G - L$ and $(xa)^2 \in U - N$.

Suppose $\eta = 1$. Then $b^{x^2} = b^{-1}z = b(a^2)$ and we apply Lemma 1.2 in the group $G/\langle a^2\rangle = \bar{G}$. We have $o(\bar{x}) = o(\bar{b}) = 4$ and $[\bar{x}^2, \bar{b}] = 1 = [\bar{x}, \bar{b}^2]$. If $[\bar{x}, \bar{b}] = 1$, then $[x, b] \in \langle a^2\rangle$ (and $\langle x, b\rangle$ is of class ≤ 2 since $\langle x, b\rangle' \leq \langle a^2\rangle$) and so $[x^2, b] = [x, b]^2 = 1$, a contradiction. Thus $[\bar{x}, \bar{b}] \neq 1$ and so (by Lemma 1.2) $o(\overline{xb}) = 2$. Hence $(xb)^2 \in \langle a^2\rangle$, contrary to the fact that $xb \in G - L$ and $(xb)^2 \in U - N$. Our claim (ii) is proved.

(iii) The factor group $G/\langle z\rangle$ ($z \in \Omega_1(Z(G))$) is not isomorphic to a Wilkens group of type (b).

Suppose false. Then $G/\langle z\rangle$ has a maximal normal elementary abelian subgroup $N/\langle z\rangle$ so that G/N is cyclic and $G/\langle z\rangle$ is not D_8-free. If N is elementary abelian, then G is a Wilkens group of type (b) with respect to N, a contradiction. Hence N is not elementary abelian and so $\mho_1(N) = \langle z\rangle$ and $G \neq N$. Set $L/N = \Omega_1(G/N)$.

(α) Suppose that N is abelian.

Then $|N : \Omega_1(N)| = 2$ and, for each $a \in N - \Omega_1(N)$, $a^2 = z$. Also, $\langle N - \Omega_1(N)\rangle = N$, $\Omega_1(G) \leq L$, and $\Omega_1(G/\langle z\rangle) \leq L/\langle z\rangle$. Let $v \in L - N$ with $v^2 = z$ and let x be any element in $N - \Omega_1(N)$. By Lemma 1.2, $[v, x] = 1$ and so v centralizes N. But then L is abelian with $\mho_1(L) = \langle z\rangle$ and L is normal in G, contrary to our assumption that $N/\langle z\rangle$ is a maximal normal elementary abelian subgroup of $G/\langle z\rangle$. We have proved that for each $v \in L - N$, $v^2 \neq z$.

Suppose that for each $x \in L - N$, $x^2 \in N - \Omega_1(N)$. If $G = \langle N, g\rangle$, then $\langle g\rangle$ covers $G/\Omega_1(N)$. But then $G/\Omega_1(N)$ is cyclic and (since G is not D_8-free) G is a Wilkens group of type (b), a contradiction. We have proved that there is $x \in L - N$ with $x^2 \in \Omega_1(N)$.

Suppose that there are no involutions in $L - N$. There is $x \in L - N$ such that $x^2 \in \Omega_1(N) - \langle z\rangle$. Suppose that x does not commute with an $a \in N - \Omega_1(N)$. By Lemma 1.2, xa is an involution, a contradiction. Hence x commutes with all elements in $N - \Omega_1(N)$ and therefore L is abelian of type $(4, 4, 2, \ldots, 2)$. We have $\Omega_1(L) = \Omega_1(N) = \Omega_1(G)$ and $L/\Omega_1(L) \cong E_4$. Since G is nonabelian, we have $L < G$. Because $N/\Omega_1(N)$ is a normal subgroup of order 2 in $G/\Omega_1(N)$ and G/N

is cyclic, $G/\Omega_1(N)$ is abelian of type $(2^s, 2), s \geq 2$. For each $y \in L - N$, $y^2 = x^2$ or $y^2 = x^2z$. Hence, if $g \in G$ is such that $G = \langle N, g\rangle$, then $\Omega_1(\langle g\rangle) = \langle x^2\rangle$ or $\Omega_1(\langle g\rangle) = \langle x^2z\rangle$ and so $\mho_1(L) = \langle z, x^2\rangle \leq Z(G)$. We may assume (by a suitable notation) that $\langle g\rangle \geq \langle x\rangle$. Set $M = \langle g\rangle\Omega_1(N)$ so that M is a maximal subgroup of G. Suppose that $G/\langle x^2\rangle$ is nonmodular so that $G/\langle x^2\rangle$ is a Wilkens group. But $\Omega_1(G/\langle x^2\rangle) = S_0/\langle x^2\rangle$, where $S_0 = \Omega_1(N)\langle x\rangle$. On the other hand, G/S_0 is noncyclic (since L/S_0 and M/S_0 are two nontrivial cyclic subgroups of G/S_0 with $(L/S_0)\cap(M/S_0) = \{1\}$). This is a contradiction and so $G/\langle x^2\rangle$ is modular. Since $\Omega_1(G/\langle z\rangle) = N/\langle z\rangle$, we may use Lemma 1.12 and we see that $G/\langle x^2, z\rangle$ is nonmodular. This is not possible since $G/\langle x^2\rangle$ is modular. We have proved that there are involutions in $L - N$.

Let t be an involution in $L - N$. If t centralizes an element $a \in N - \Omega_1(N)$, then $ta \in L - N$ and $(ta)^2 = z$, a contradiction. Thus, $C_N(t) \leq \Omega_1(N)$. For any $x \in L - N$, $C_N(x) = C_N(t)$ and so $x^2 \in \Omega_1(N)$. It follows that $\exp(L) = 4$.

Suppose that t does not centralize $\Omega_1(N)$. Let w be a fixed involution in $\Omega_1(N) - C_N(t)$ and let a be a fixed element in $N - \Omega_1(N)$. We have $w^t = wu$ with $1 \neq u \in C_N(t)$ and

$$(tw)^2 = (twt)w = wuw = u,$$

and so $u \in C_N(t) - \langle z\rangle$ (since $tw \in L - N$). Since $C_N(tw) = C_N(t) < \Omega_1(N)$, we have $[tw, a] \neq 1$. By Lemma 1.2, $(tw)a$ is an involution. We get

$$1 = (twa)^2 = twatwa = w^t a^t wa = wua^t wa = uaa^t,$$

and so $a^t = a^{-1}u = aa^2u = a(uz)$. We consider the factor group $L/\langle uz\rangle = \bar{L}$ and we see that $o(\bar{a}) = 4$, $o(\bar{t}) = 2$, $[\bar{a}, \bar{t}] = 1$, $\bar{a}^2 = \bar{z}$. Also, $w^t = wu = wz(uz)$ gives $\bar{w}^{\bar{t}} = \bar{w}\bar{z}$ so that $\langle \bar{w}, \bar{t}\rangle \cong D_8$ with $Z(\langle \bar{w}, \bar{t}\rangle) = \langle \bar{z}\rangle$. Hence $\langle \bar{w}, \bar{t}\rangle\langle \bar{a}\rangle$ is the central product $D_8 * C_4$, contrary to Lemma 1.1 (applied in $\bar{L}$). We have proved that t centralizes $\Omega_1(N)$ so that $S = \langle t\rangle \times \Omega_1(N)$ is an elementary abelian maximal subgroup of L. Since L is nonabelian and $\exp(L) = 4$, L is nonmodular.

Let $a \in N - \Omega_1(N)$. Then by Lemma 1.13, $a^t = a^{-1}s$, $s \in \Omega_1(N)$. If $s = 1$, then t acts invertingly on N, G/N is cyclic and so G would be a Wilkens group of type (a), a contradiction. Thus, $s \neq 1$ and $(ta)^2 = (tat)a = a^{-1}sa = s$ and so $s \in \Omega_1(N) - \langle z\rangle$. Set $ta = y$ so that $y^2 = s$ and, since $C_N(y) = C_N(t)$, we have for each $n \in C_N(t) = \Omega_1(N)$, $(ny)^2 = n^2y^2 = s$ and $[t, N] = \langle zs\rangle = L'$. The group L has exactly three abelian maximal subgroups: $S = \Omega_1(G)$, N, and $\Omega_1(N)\langle y\rangle$, where only S is elementary abelian and all three are normal in G

with $\mho_1(\Omega_1(N)\langle y\rangle) = \langle s\rangle$, $s \in \Omega_1(N) - \langle z\rangle$. We have $\Omega_1(N) = Z(L)$ and z, s and zs are central involutions in G.

If $G = L$, then G would be a Wilkens group of type (b), a contradiction. Thus $L < G$. Since G/N is cyclic of order $2^s, s \geq 2$, we have $G = \langle N, g\rangle$ with $g^{2^{s-1}} \in L - N$. If $g^{2^{s-1}} \in L - N - S$, then $\langle g\rangle$ covers G/S, G/S is cyclic and so G would be a Wilkens group of type (b) with respect to S, a contradiction. It follows that $g^{2^{s-1}} \in S - \Omega_1(N)$. If L is not maximal in G, then there is a subgroup $K > L$ with $|K : L| = 2$ and so K is a Wilkens group. Since $S = \Omega_1(K)$ and $K/S \cong E_4$, we have a contradiction. Indeed, $(S\langle g^{2^{s-2}}\rangle)/S$ and L/S are two distinct subgroups of order 2 in K/S. Hence $|G : L| = 2$ and for each $g \in G - L$, $g^2 \in S - \Omega_1(N)$.

We apply now Lemma 1.2 in the group $G/\langle zs\rangle = \bar{G}$, where $\langle zs\rangle = L'$. We have for some elements $g \in G - L$ and $a \in N - \Omega_1(N)$, $o(\bar{g}) = o(\bar{a}) = 4$ and, since $a^{g^2} = a(zs)$, we get $[\bar{a}, \bar{g}^2] = 1 = [\bar{a}^2, \bar{g}]$. If $[\bar{g}, \bar{a}] = 1$, then $[g, a] \in \langle zs\rangle$ and so $[g^2, a] = [g, a]^2 = 1$, a contradiction. Hence $[\bar{g}, \bar{a}] \neq 1$ and so (by Lemma 1.2) $o(\bar{g}\bar{a}) = 2$ which gives $(ga)^2 \in \langle zs\rangle$, contrary to $ga \in G - L$ and (by the above) $(ga)^2 \in S - \Omega_1(N)$. We have proved that N must be nonabelian.

(β) Suppose that N is nonabelian.

This case is very difficult. We have $\mho_1(N) = N' = \langle z\rangle$. Let D be a minimal nonabelian subgroup of N so that $D' = \langle z\rangle$. Since $d(D) = 2$ and $D/\langle z\rangle$ is elementary abelian, we have $D/\langle z\rangle \cong E_4$ and therefore $D \cong D_8$. The subgroup D is normal in N and, since $D' = N'$, $C_N(D)$ covers N/D. By Lemma 1.1, $C_N(D)$ is elementary abelian. We have $C_N(D) = Z(N)$, $N = DZ(N)$ with $D \cap Z(N) = Z(D) = \langle z\rangle$ so that $Z(N)$ is normal in G. Set

$$D = \langle a, u | a^4 = u^2 = 1, a^2 = z, a^u = a^{-1}\rangle$$

so that $A = \langle a, Z(N)\rangle$, $E_1 = \langle u, Z(N)\rangle$, and $E_2 = \langle au, Z(N)\rangle$ are all abelian maximal subgroups of N, where A is of exponent 4 (all elements in $A - Z(N)$ are of order 4, $\mho_1(A) = \langle z\rangle$), and E_1 and E_2 are both elementary abelian. Thus A is normal in G. All elements in $N - A$ are involutions, $Z(N) = \Omega_1(A)$, N is a Wilkens group of type (a) with respect to A and also a Wilkens group of type (b) with respect to E_1 and E_2.

Since N/A is a normal subgroup of order 2 in G/A and G/N is cyclic, it follows that G/A is abelian. If G/A were cyclic, then we have $G = \langle A, g\rangle$ with some $g \in G$ and, since $\Omega_1(\langle g\rangle)$ is of order 2 and acts invertingly on A (of exponent > 2), G is a Wilkens group of type (a), a contradiction. It follows that G/A is abelian of type $(2^m, 2), m \geq 1$ and $L/A \cong E_4$, where $L/N = \Omega_1(G/N)$. Also note that $N/Z(N) \cong E_4$.

Suppose that for each $l \in L - N$, $l^2 \in A - Z(N)$. This is equivalent to assuming that $L/Z(N) \cong C_4 \times C_2$, which also implies that both E_1 and E_2 are normal in L. If $G = \langle N, g \rangle$, then either $E_1^g = E_2$ (in which case $G > L$ and $G/Z(N) \cong M_{2^n}, n \geq 4$ and G is a Wilkens group of type (c)) or $E_1^g = E_1$, E_1 is normal in G, $\langle g \rangle$ covers G/E_1 and, since G is not D_8-free, G is a Wilkens group of type (b). In both cases we have a contradiction.

We have proved that $L/Z(N)$ is not isomorphic to $C_4 \times C_2$ and so $L/Z(N)$ is either elementary abelian or $L/Z(N) \cong D_8$. In any case, there is $l \in L - N$ with $l^2 \in Z(N)$.

(β1) Suppose that $L/Z(N) \cong D_8$.

Then $E_1^x = E_2$ for $x \in L - N$, and so $G = L$. Indeed, the cyclic group G/N acts on the set $\{E_1, E_2\}$ and so, if $G > L$, then L would normalize E_1.

Suppose in addition that there are no involutions in $L - N$. Let $l \in L - N$ with $l^2 \in Z(N)$ so that $l^2 \neq 1$ and $o(l) = 4$. Let a_0 be any element in $A - Z(N)$ so that $a_0^2 = z$. If $[l, a_0] \neq 1$, then la_0 is an involution in $L - N$, a contradiction. Thus l centralizes each element in $A - Z(N)$ and, since $\langle A - Z(N) \rangle = A$, $A\langle l \rangle$ is an abelian maximal subgroup of $G = L$. Also, $Z(N) = Z(G)$ (since $Z(L/Z(N)) = A/Z(N)$) and so we may use the relation $|G| = 2|G'||Z(G)|$, which gives $|G'| = 4$. But G' covers $A/Z(N) = (G/Z(N))'$ and so $G' \cong C_4$ is a cyclic subgroup of order 4 in A inverted by u (since u acts invertingly on A) and so we have $G'\langle u \rangle \cong D_8$. We may assume $G' = \langle a \rangle < D \cong D_8$ so that D is normal in G. By Lemma 1.1, $C_G(D)$ is elementary abelian and so $C_G(D)$ cannot cover G/N (since there are no involutions in $G - N$). Hence $C_G(D) = Z(N)$ and so $l \in G - N$ induces an outer automorphism on D. Hence $a^l = a^{-1}$, contrary to Lemma 1.1.

We have proved that there are involutions in $G - N$ and let t be one of them so that $\Omega_1(G) = G$. Since $G/Z(N) \cong D_8$, there is $k \in G - N$ such that $k^2 \in A - Z(N)$. Our ultimate goal is to show that $Z(N) = Z(G)$.

Suppose $Z(N) \neq Z(G)$. Assume there is $s \in G - N$ with $s^2 = z$ and let $a_0 \in A - Z(N)$. If $[s, a_0] \neq 1$, then (Lemma 1.2) $[s, a_0] = s^2 a_0^2 = zz = 1$, a contradiction. Thus s centralizes $A - Z(N)$ and $\langle A - Z(N) \rangle = A$ and so $Z(N) \leq Z(G)$. Since $Z(G/Z(N)) = A/Z(N)$, we have $Z(N) = Z(G)$. This is a contradiction and so there is no $s \in G - N$ with $s^2 = z$. In particular, the involution t does not centralize any element in $A - Z(N)$.

Now, $A\langle t \rangle$ is a maximal subgroup of G. If x is any element of order 8 in At, then $o(x^2) = 4$ and $x^2 \in A - Z(N)$. But then t centralizes x^2 (noting that $C_A(t) = C_A(x)$), a contradiction. Hence $\exp(A\langle t \rangle) = 4$ and, since $A\langle t \rangle$ is nonabelian, $A\langle t\rangle$ is nonmodular and therefore $A\langle t\rangle$ must be a Wilkens group. If X is an abelian maximal subgroup of $A\langle t\rangle$ distinct from A, then $X \cap A \le Z(N)$ (recalling that t does not commute with any element in $A - Z(N)$) and, since $|A : (X \cap A)| = 2$, we get $X \ge Z(N)$ and so $Z(N) = Z(G)$, a contradiction. Hence A is the unique abelian maximal subgroup of $A\langle t\rangle$.

If all elements in $(A\langle t\rangle) - A$ are involutions, then t inverts each element of A and so t centralizes $Z(N)$ and then $Z(N) = Z(G)$, a contradiction. It follows that there is an element c of order 4 in At such that $c^2 \in Z(N) - \langle z\rangle$. But $[c, a] \ne 1$ (since t does not centralize a) and so (by Lemma 1.2) ac is an involution for each $a \in A - Z(N)$. Since t does not centralize $Z(N)$ (otherwise $Z(N) = Z(G)$), $Z(N)\langle t\rangle$ contains less than $2|Z(N)| - 1$ involutions. All $|Z(N)|$ elements ca ($a \in A - Z(N)$) are involutions and so $A\langle t\rangle$ contains at least $2|Z(N)| - 1$ involutions. This shows that $\Omega_1(A\langle t\rangle) = A\langle t\rangle$. Since A (of exponent > 2) is the unique abelian maximal subgroup of $A\langle t\rangle$, $A\langle t\rangle$ must be a Wilkens group of type (a). In that case t acts invertingly on A and so t centralizes $Z(N)$ and $Z(N) = Z(G)$, a contradiction.

We have proved that $Z(N) = Z(G)$ and so $Z(N)\langle k\rangle = A\langle k\rangle$ is an abelain maximal subgroup of G (noting that $k^2 \in A - Z(N)$ and so $Z(N)\langle k\rangle/Z(N)$ is cyclic). Using the relation $|G| = 2|G'||Z(G)|$ we get $|G'| = 4$ and so $G' \cong C_4$ (since G' covers $A/Z(N)$). We may assume (as before) that $G' \le D$ and so D is normal in G with $C_G(D) = Z(N)$ (since $C_G(D)$ is elementary abelian and $|G : Z(G)| = 8$). The involution t induces an outer automorphism on D and so $D\langle t\rangle \cong D_{2^4}$. It follows that $G = (D\langle t\rangle) \times E_{2^m}$ for some $m \ge 1$, which is a Wilkens group of type (a), a contradiction.

($\beta 2$) We have proved that we must have $L/Z(N) \cong E_8$ and so $\exp(L) = 4$.

In that case we prove first that there are involutions in $L - N$. Suppose false. If v is any element in $L - N$ and $a_0 \in A - Z(N)$, then $1 \ne v^2 \in Z(N)$ and $a_0^2 = z$ so that we may apply Lemma 1.2. If $[v, a_0] \ne 1$, then va_0 is an involution, a contradiction. Hence $L = \langle L - N\rangle$ centralizes $A = \langle A - Z(N)\rangle$. In particular, $A \le Z(N)$, which contradicts the fact that $Z(N) < A$. We have proved that there are involutions in $L - N$ and so $L = \Omega_1(L) = \Omega_1(G)$ (noting that G/N is cyclic and $L/N = \Omega_1(G/N)$).

Assume $C_G(D) > Z(N)$ so that $C_G(D)$ covers L/N and (Lemma 1.1) $C_G(D)$ is elementary abelian. In that case, $L \cong D_8 \times E_{2^m}$ and $\mho_1(L) = \langle z\rangle$, which contradicts our assumption that $N/\langle z\rangle$ is a maximal normal elementary abelian subgroup of $G/\langle z\rangle$. Hence $C_G(D) = Z(N)$. If D were normal in L, then $L/Z(N) \cong D_8$ since $\mathrm{Aut}(D) \cong D_8$. This is a contradiction and so $N_G(D) = N$.

In particular, $Z(L) \leq Z(N)$ and $Z(N) > \langle z \rangle$.

(β2a) We assume that $G > L$.

Hence $L = \Omega_1(L)$ (being nonmodular) is a Wilkens group of type (a) or (b). In particular, L possesses an abelian maximal subgroup B. Since $B \cap N$ is an abelian maximal subgroup of N, we get $B \cap N \in \{A, E_1, E_2\}$. Hence $B \cap N \geq Z(N)$ and so $Z(N) \leq Z(L)$. By the result in the previous paragraph, we get $Z(N) = Z(L)$. Since $|L : Z(L)| = 8$, B is the unique abelian maximal subgroup of L and so B is normal in G. Using the relation $|L| = 2|L'||Z(L)|$, we get $|L'| = 4$. Since $L' \leq Z(L)$, $L' \cong E_4$ and $L' > \langle z \rangle$.

It is easy to see that $|G : L| = 2$. Suppose false. Let $K < G$ be such that $|K : L| = 2$. Since $L = \Omega_1(K)$, K must be a Wilkens group of type (a) or (b). By the uniqueness of B in L, it follows that K/B is a cyclic group (of order 4). Since $N < L$ and G/N is cyclic, G/L is cyclic. But L/B is a normal subgroup of order 2 in G/B and so G/B is abelian. We have $L/B \leq \Omega_1(G/B)$. On the other hand, G/L is cyclic and $K/L = \Omega_1(G/L)$ so that $\Omega_1(G/B) \leq K/B$. Since $K/B \cong C_4$, we get $\Omega_1(G/B) \leq L/B$ and so $\Omega_1(G/B) = L/B$. Hence G/B is cyclic. If L is of type (a) (in the case $\exp(B) > 2$), then all elements in $L - B$ are involutions and so G is also a Wilkens group of type (a), a contradiction. If L is of type (b), then B must be elementary abelian and (since G is not D_8-free) G is also a Wilkens group of type (b), a contradiction. We have proved that $|G : L| = 2$ and $G/B \cong E_4$ (because in case $G/B \cong C_4$, G would be a Wilkens group of type (a) or (b)). But $G/N \cong C_4$ and so for each $x \in G - L$, $x^2 \in B - N$.

We assume first that $\exp(B) > 2$. Then L is a Wilkens group of type (a). In that case $B/Z(N) \cong E_4$, all elements in $L - B$ are involutions and $\Omega_1(B) = Z(N)$. Indeed, if $|B : \Omega_1(B)| = 2$, then acting (invertingly) with an involution in $L - B$ on B, we see that $|L'| = 2$, a contradiction. Hence we must have $\Omega_1(B) = Z(N)$ so that B is abelian of type $(4, 4, 2, \ldots, 2)$, $B \cap N = A$, u acts invertingly on B, each element in $B - N$ is of order 4, and if $k \in B - N$, then $k^2 = z_0 \in L' - \langle z \rangle$ (noting that $k^u = k^{-1}$ and so $k^2 \in L'$) and $L'\langle a, k \rangle = \langle a, k \rangle \cong C_4 \times C_4$ (if $k^2 = z$, then ak would be an involution in $B - N$, contrary to $\Omega_1(B) = Z(N)$). Let $x \in G - L$ so that $x^2 \in B - N$ and we may assume that $x^2 = k$, where $k^2 = z_0 \in L' - \langle z \rangle$. In particular, $C_G(z_0) \geq \langle L, x \rangle = G$ and so $L' \leq Z(G)$. Let x' be an arbitrary element in $G - L$ so that $(x')^2 = k' \in B - N$ and $(k')^2 = z' \in L' - \langle z \rangle$. Consider the factor group $\bar{G} = G/\langle z' \rangle$. We have $o(\bar{a}) = o(\overline{x'}) = 4$ and $[\bar{a}^2, \overline{x'}] = 1 = [\bar{a}, (\overline{x'})^2]$ and so we may use Lemma 1.2. If $[\bar{a}, \overline{x'}] \neq 1$, then $o(\bar{a}\overline{x'}) = 2$ and so $(ax')^2 \in \langle z' \rangle \leq N$, a contradiction. Hence we must have $[\bar{a}, \overline{x'}] = 1$ and so $[a, x'] \in \langle (x')^4 \rangle$. It follows that $[a, x] = z_0^\epsilon$ ($\epsilon = 0, 1$).

Consider the element $y = xu \in G - L$ and compute

$$[a, y] = [a, xu] = [a, u][a, x]^u = zz_0^\epsilon \neq 1.$$

On the other hand, $[a, y] \in \langle y^4 \rangle$, $y^4 \in L' - \langle z \rangle$. Thus $\epsilon = 1$ and $y^4 = zz_0$. Finally, consider the factor group $\tilde{G} = G/\langle zz_0 \rangle$ so that $o(\tilde{y}) = o(\tilde{k}) = 4$ and $[\tilde{y}, \tilde{k}^2] = 1 = [\tilde{y}^2, \tilde{k}]$ and apply again Lemma 1.2. Since $[k, x] = 1$, we get

$$[k, y] = [k, xu] = [k, u][k, x]^u = k^2 = z_0,$$

and so $[\tilde{k}, \tilde{y}] \neq 1$. Thus $o(\tilde{k}\tilde{y}) = 2$ and so $(ky)^2 \in \langle zz_0 \rangle \leq N$. This is a contradiction since $ky \in G - L$.

We study now the case $\exp(B) = 2$ and L is a Wilkens group of type (b). We may assume that $B \cap N = E_1$. Since $G/B \cong E_4$, $\exp(G) = 4$. Let t be an involution in $B - N$. Since $B \geq E_1 = Z(N)\langle u \rangle$, t centralizes $u \in D$. But $N_L(D) = N$ and so $[a, t] \in L' - \langle z \rangle$. Suppose that $L' \leq Z(G)$. Let $x \in G - L$ so that $x^2 \in B - N$ and (by the above) $[a, x^2] = z_0 \in L' - \langle z \rangle$. Consider the factor group $G/\langle z_0 \rangle = \bar{G}$ so that $o(\bar{a}) = o(\bar{x}) = 4$. If $[\bar{x}, \bar{a}] = 1$, then $[x, a] \in \langle z_0 \rangle$. But $\langle x, a \rangle' = \langle z_0 \rangle$ and so $\langle x, a \rangle$ is of class 2. Thus $[x^2, a] = [x, a]^2 = 1$, a contradiction. Hence $[\bar{x}, \bar{a}] \neq 1$ and so (by Lemma 1.2) $o(\bar{x}\bar{a}) = 2$, which gives $(xa)^2 \in \langle z_0 \rangle$, contrary to $xa \in G - L$ and $(xa)^2 \in B - N$. We have proved that $L' \not\leq Z(G)$. Again, let $x \in G - L$ so that $\langle x \rangle$ induces an involutory automorphism on $Z(L) = Z(N)$. Suppose $Z(N) > L'$. Then there is an involution $z' \in Z(L) - L'$ centralized by x and so $z' \in Z(G)$. Since $G/\langle z' \rangle$ is nonmodular (noting that $D \cap \langle z' \rangle = \{1\}$), $G/\langle z' \rangle$ is a Wilkens group. All squares of elements of G lie either in $B - N$ or in L'. Indeed, let $as (s \in B)$ be any element in $L - B$. Then $(as)^2 = a^2(a^{-1}sas) = z[a, s] \in L'$. Therefore z' is not a square of any element in G, which implies $L/\langle z' \rangle = \Omega_1(G/\langle z' \rangle)$. Let $y \in L$ be such that $[y, L] \leq \langle z' \rangle$. But $\langle z' \rangle \cap L' = \{1\}$ and so $[y, L] = \{1\}$ and therefore $y \in Z(L)$. Hence $Z(L/\langle z' \rangle) = Z(L)/\langle z' \rangle$. Since $|L : Z(L)| = 8$, the elementary abelian group $B/\langle z' \rangle$ is the unique abelian maximal subgroup of $L/\langle z' \rangle$. It follows that $G/\langle z' \rangle$ must be a Wilkens group of type (b) with respect to $B/\langle z' \rangle$. But then G/B must be cyclic, a contradiction. We have proved that $L' = Z(L)$ and so $|G| = 2^6$. Now, $A = \langle a \rangle \times \langle z_0 \rangle = \langle a \rangle L' \cong C_4 \times C_2$ is normal in G and is self-centralizing in L (since B is the unique abelian maximal subgroup of L). If $C_G(A) \not\leq L$, then there is $g \in C_G(A) - L$ such that $g^2 \in A \leq N$, contrary to $g^2 \in B - N$. Thus A is self-centralizing in G and so $G/A \cong D_8$ since $\mathrm{Aut}(A) \cong D_8$. On the other hand, N/A is a normal subgroup of order 2 in G/A and $G/N \cong C_4$ so that G/A is abelian. This is a contradiction. We have proved that the case $G > L$ is not possible.

(β2b) It remains to study the case $G = L = \Omega_1(G)$. Since $G/Z(N) \cong E_8$, $\exp(G) = 4$. If for each $x \in G - N$, $x^2 \in \langle z \rangle$, then $G/\langle z \rangle$ is elementary abelian, contrary to our assumption that $N/\langle z \rangle$ is a maximal normal elementary abelian subgroup of $G/\langle z \rangle$. Hence there is $k \in G - N$ such that $k^2 \in Z(N) - \langle z \rangle$. It follows that $k^2 \in Z(G)$ and so $\langle z \rangle < Z(G) \leq Z(N)$.

Let $z' \in Z(G) - \langle z \rangle$. Since $G/\langle z' \rangle$ is nonmodular (noting that $D \cap \langle z' \rangle = \{1\}$ with $D \cong D_8$), it follows that $G/\langle z' \rangle$ is a Wilkens group with $\Omega_1(G/\langle z' \rangle) = G/\langle z' \rangle$ and so $G/\langle z' \rangle$ cannot be of type (c). Hence $G/\langle z' \rangle$ must be of type (b) by our previous result (ii). Thus, G has a maximal subgroup N_1 containing z' such that $N_1/\langle z' \rangle$ is a maximal normal elementary abelian subgroup of $G/\langle z' \rangle$. By our previous result (iii)(α), $\langle z' \rangle = \mho_1(N_1) = N_1'$ (since N_1 must be nonabelian). In particular, z' is a square of an element in $G - N$ and z' is a commutator in G. Conversely, if $k \in G - N$, then $k^2 \in Z(G)$ so that $\Phi(G) \leq Z(G)$. Hence $Z(G) = G' = \Phi(G)$ and so G is a special group. Now, $N \cap N_1$ is a maximal subgroup of N and, since $\mho_1(N \cap N_1) \leq \langle z \rangle \cap \langle z' \rangle = \{1\}$, $N \cap N_1$ is elementary abelian and so $N \cap N_1 = E_1$ (or E_2) containing $Z(N)$ and (by the structure of $N_1 \cong N$) $Z(N_1)$ is a subgroup of index 2 in $N \cap N_1$. But $Z(N) \cap Z(N_1) \leq Z(G)$ and so $|Z(N) : Z(G)| \leq 2$.

Suppose that $|Z(N) : Z(G)| = 2$ and let s be an involution in $Z(N) - Z(G)$. Let t be an involution in $G - N$. We have $[t, s] = s_0$ with $s_0 \in Z(G)$ and $s_0 \neq 1$ since $Z(N) \neq Z(G)$. If $n \in N$, then

$$[tn, s] = [t, s][n, s] = [t, s] = s_0 \in Z(G).$$

Hence, for each $x \in G - N$, $[x, s] = s_0$, where s_0 is a fixed involution in $Z(G)$. Take an involution $z' \in Z(G) - \langle z \rangle$. Let N_1 be a maximal subgroup of G such that $\langle z' \rangle = \mho_1(N_1) = N_1'$. Then $N \cap N_1$ is an elementary abelian maximal subgroup of N containing $Z(N)$. If $f_1 \in N_1 - N$, then $[f_1, s] = z' = s_0$. Let N_2 ($\neq N_1$) be a maximal subgroup of G such that $\langle zz' \rangle = \mho_1(N_2) = N_2'$. Then again, $N \cap N_2 \geq Z(N)$ and, if $f_2 \in N_2 - N$, then $[f_2, s] = zz' = s_0$. Hence $zz' = z'$ and so $z = 1$, a contradiction.

We have proved that $Z(N) = Z(G)$. Suppose that $Z(G)$ possesses a four-subgroup $\langle z_1, z_2 \rangle$ such that $\langle z_1, z_2 \rangle \cap \langle z \rangle = \{1\}$ (which is equivalent to the assumption $|Z(G)| \geq 8$). Let $k_1, k_2 \in G - N$ be such that $k_1^2 = z_1$ and $k_2^2 = z_2$. Suppose that k_1 centralizes all elements (of order 4) in $A - Z(N)$. Then k_1 centralizes $A = Z(N)\langle a \rangle = \langle A - Z(N) \rangle$ and so $A\langle k_1 \rangle$ is an abelian maximal subgroup of G. Using a result of A. Mann (Lemma 1.5 with respect to maximal subgroups $A\langle k_1 \rangle$ and N), we get $|G'| \leq 4$. But $G' = Z(G)$ is of order ≥ 8, a contradiction. We may assume that $[a, k_1] \neq 1$ and so (Lemma 1.2) $[a, k_1] =$

$a^2k_1^2 = zz_1$. We have either $[a, k_2] = 1$ or $[a, k_2] \neq 1$, in which case (Lemma 1.2) $[a, k_2] = a^2k_2^2 = zz_2$. We compute

$$[a, k_1k_2] = [a, k_1][a, k_2] = zz_1[a, k_2]$$

and so either $[a, k_1k_2] = zz_1$ or $[a, k_1k_2] = zz_1zz_2 = z_1z_2$. But the element k_1k_2 is contained in N and so $[a, k_1k_2] \in \langle z \rangle$, a contradiction.

We have proved that we must have $Z(N) = Z(G) \cong E_4$ and so $|G| = 2^5$. Let $z' \in Z(G) - \langle z \rangle$ and let $k_1, k_2 \in G - N$ be such that $k_1^2 = z'$ and $k_2^2 = zz'$. Suppose $[k_1, k_2] = 1$ so that $\langle k_1, k_2 \rangle \cong C_4 \times C_4$, $k_1k_2 \in N$ and $(k_1k_2)^2 = k_1^2k_2^2 = z$, and therefore we may assume that $k_1k_2 = a$. Hence $\langle k_1, k_2 \rangle$ is an abelian maximal subgroup of G normalized by u, where u inverts each element in

$$\langle k_1, k_2 \rangle \cap N = \langle a \rangle \times \langle z' \rangle \cong C_4 \times C_2 \quad \text{and} \quad G = \langle k_1, k_2 \rangle \langle u \rangle.$$

We know that there are involutions in $G - N$ and so there is an element $k \in \langle k_1, k_2 \rangle - N$ such that uk is an involution. This gives $(uk)^2 = ukuk = 1$, $k^u = k^{-1}$ and so u inverts each element of the abelian group $\langle k_1, k_2 \rangle$. In this case G is a Wilkens group of type (a), a contradiction. Hence $[k_1, k_2] \neq 1$ and, using Lemma 1.2, we get $[k_1, k_2] = k_1^2k_2^2 = z'(zz') = z$, $o(k_1k_2) = 2$, $k_1k_2 \in N$. Because $k_1k_2 \notin Z(\langle k_1, k_2 \rangle)$, we may assume $k_1k_2 = u$ (an involution in $N - A$). Since $[k_1, u] = [k_1, k_1k_2] = [k_1, k_2] = z$ (which gives $u^{k_1} = uz$) and $D = \langle a, u \rangle \cong D_8$ is not normal in G, we have $[a, k_1] \neq 1$. By Lemma 1.2, $[a, k_1] = a^2k_1^2 = zz'$ and k_1a is an involution in $G - N$. We compute

$$u^{k_1a} = (uz)^a = (uz)z = u.$$

Hence $\langle u, k_1a \rangle \cong E_4$ with $\langle u, k_1a \rangle \cap Z(G) = \{1\}$ and so $\langle u, k_1a, Z(G) \rangle \cong E_{16}$. Since G is not D_8-free, it follows that G is a Wilkens group of type (b). This is our final contradiction and so our statement (iii) is completely proved.

(iv) The factor group $G/\langle z \rangle$ ($z \in \Omega_1(Z(G))$) is not isomorphic to a Wilkens group of type (c).

Suppose false. Then $G/\langle z \rangle$ is a Wilkens group of type (c). Hence we may set $\Omega_1(G/\langle z \rangle) = H/\langle z \rangle$ (implying that $\Omega_1(G) \leq H$) so that $H = H_1H_2$, where H_1 and H_2 are normal subgroups in H with $H_1 \cap H_2 = \langle z \rangle$, $H_1/\langle z \rangle \cong D_8$, $H_2/\langle z \rangle$ is elementary abelian and G/H is cyclic of order ≥ 4. Let $Z/\langle z \rangle$ be the unique cyclic subgroup of index 2 in $H_1/\langle z \rangle$ and set $Z(H_1/\langle z \rangle) = Z_0/\langle z \rangle$ so that $|Z : Z_0| = 2$ and $|Z_0 : \langle z \rangle| = 2$. Let $Z_0H_2 = N$, $ZH_2 = A$, so that $Z(H/\langle z \rangle) = N/\langle z \rangle$ and $N/\langle z \rangle$ is the unique maximal normal elementary abelian subgroup of $G/\langle z \rangle$, $G/N \cong M_{2^n}, n \geq 4$, $A/N = (G/N)'$, $H/N = \Omega_1(G/N) \cong E_4$. If E_1/N

and E_2/N are another two subgroups of order 2 in H/N (distinct from A/N), then $E_1^G = E_2$, $E_1/\langle z\rangle$ and $E_2/\langle z\rangle$ are elementary abelian and $A/\langle z\rangle$ is abelian of type $(4, 2, \dots, 2)$. Also, $N_G(E_1) = N_G(E_2) > H$ and $|G : N_G(E_1)| = 2$. Finally, G possesses a subgroup S such that $G = HS$, $H \cap S = Z$, and $S/\langle z\rangle$ is cyclic so that S is abelian. In fact, S is either cyclic or abelian of type $(2^n, 2), n \geq 4$. Also, S is cyclic if and only if Z_0 is cyclic (since $Z_0/\langle z\rangle = \Omega_1(S/\langle z\rangle)$). We have $H' \leq Z_0$ and H' covers $Z_0/\langle z\rangle$.

(α) Suppose Z_0 (and so also S) is cyclic.

We have $H_1/\langle z\rangle \cong D_8$ and $(H_1/\langle z\rangle)' = Z_0/\langle z\rangle$. Since H_1' covers $Z_0/\langle z\rangle$ and Z_0 is cyclic, we get that $H_1' = Z_0$ is of index 4 in H_1. By a very well known result of O. Taussky, H_1 is of maximal class. Since H_1 is Q_8-free, we get $H_1 \cong D_{2^4}$. An involution $t \in H_1 - Z$ inverts Z and so $H_3 = \langle Z_0, t\rangle \cong D_8$. The subgroup H_3 is normal in H_1 and $[H_3, H_2] \leq H_1 \cap H_2 = \langle z\rangle$ implies that H_3 is normal in H. Since $C_{H_1}(H_3) = \langle z\rangle$ and $H_1/\langle z\rangle \cong D_8 \cong \mathrm{Aut}(H_3)$, we get that $C_H(H_3)$ covers H/H_1. By Lemma 1.1, $C_H(H_3)$ is elementary abelian. In particular, $\Omega_1(H) = H = \Omega_1(G)$. Since H is nonmodular, H is a Wilkens group with $\Omega_1(H) = H$ and so H is of type (a) or (b). It follows that H has an abelian maximal subgroup $\tilde{A}$ which is unique (by a result of A. Mann) since $H' \geq H_1'$ and $|H_1'| = 4$ (see Lemma 1.5). In particular, $\tilde{A}$ is normal in G. Also, $\tilde{A} \cap H_1 = Z \cong C_8$ since Z is the unique abelian maximal subgroup of $H_1 \cong D_{2^4}$ and so $\exp(\tilde{A}) > 2$. If $t \in H_1 - Z$, then H must be a Wilkens group of type (a) and the involution t acts invertingly on $\tilde{A}$. Let $\tilde{N} = \Omega_2(\tilde{A}) = Z_0\Omega_1(\tilde{A})$ (since $\tilde{A}/Z$ is elementary abelian) so that $\tilde{N}$ is normal in G, and $\tilde{N}/\langle z\rangle$ is a normal elementary abelian subgroup of $G/\langle z\rangle$. By the uniqueness of $N/\langle z\rangle$, $\tilde{N} \leq N$ and so $\tilde{A} = Z\tilde{N} \leq ZN = A$ and therefore $\tilde{A} = A$ is abelian of type $(8, 2, \dots, 2)$. Let $G_1 > H$ be such that $|G_1 : H| = 2$. Hence $G_1 \neq G$ and $G = HS_1$, where $S_1 = S \cap G_1$. It follows that G_1 is also a Wilkens group with $\Omega_1(G_1) = H$. Since $|G_1 : H| < 4$, G_1 is of type (a) or (b). But $A = \tilde{A}$ is the unique abelian maximal subgroup of H and $\exp(A) > 2$. Thus G_1 must be a Wilkens group of type (a) with respect to A and so $G_1/A \cong C_4$. On the other hand, we know that $S_1 \cap H = Z$. Hence, if $x \in S_1 - H$, then $x^2 \in Z \leq A$. This is a contradiction since $G_1/A \cong C_4$ and therefore $x^2 \in H - A$.

(β) Suppose $Z_0 \cong E_4$ and so S splits over $\langle z\rangle$.

We set $Z = \langle a\rangle \times \langle z\rangle$ so that $Z_0 = \langle a^2\rangle \times \langle z\rangle$ and, replacing a with az (if necessary), we may put $S = \langle s\rangle \times \langle z\rangle$ with $\Omega_2(\langle s\rangle) = \langle a\rangle = S \cap H$. If $|H_1'| = 4$, then $|H_1 : H_1'| = 4$, $H_1' = Z_0$, and (by a result of O. Taussky) H_1 is of maximal class. In that case Z_0 would be cyclic, a contradiction.

We have proved that $|H_1'| = 2$ and so $Z_0 = H_1' \times \langle z \rangle$. We set $H_1' = \langle z_0 \rangle$ so that z_0 is a central element in H. But S is abelian, $G = HS$, and $S \cap H = Z > Z_0 = \langle z, z_0 \rangle$. It follows $C_G(\langle z, z_0 \rangle) \geq \langle H, S \rangle = G$ and so $\langle z, z_0 \rangle \leq Z(G)$.

We show that there are exactly two possibilities for the structure of H_1. Since $H_1/Z_0 \cong E_4$, we have $\exp(H_1) = 4$. Suppose that H_1 is minimal nonabelian. If H_1 is metacyclic (of exponent 4), then we know that H_1 is not Q_8-free. Thus H_1 must be nonmetacyclic and we know that there is only one such minimal nonabelian group of order 2^4 and exponent 4. In particular, there is an element $b \in H_1 - Z$ such that $b^2 = z$, $[a,b] = a^2b^2 = a^2z = z_0$, where $H_1' = \langle z_0 \rangle$, z_0 is not a square in H_1, and ab is an involution so that $\Omega_1(H_1) = \langle z, z_0, ab \rangle$. Suppose now that H_1 is not minimal nonabelian. Then there is a subgroup $D \cong D_8$ in H_1 which covers $H_1/\langle z \rangle$. Thus $H_1 = D \times \langle z \rangle$ and $H_1' = D' = \langle z_0 \rangle$. We have $D \cap Z = \langle a \rangle$ or $D \cap Z = \langle az \rangle$ and all elements in $H_1 - Z$ are involutions acting invertingly on $\langle a, z \rangle$. Replacing a with az (if necessary), we may assume that $D \cap Z = \langle a \rangle$. If t is an involution in $D - Z$, then $D = \langle a, t \rangle$, where $z_0 = a^2$ is a square in H_1.

Suppose that H_2 is nonabelian. Then $H_2' = \langle z \rangle$ since $H_2/\langle z \rangle$ is elementary abelian. Let H_4 be a minimal nonabelian subgroup of H_2 so that $H_4' = \langle z \rangle$, $H_4/\langle z \rangle$ is elementary abelian and $d(H_4) = 2$. Thus $H_4/\langle z \rangle \cong E_4$ and so $H_4 \cong D_8$. The subgroup H_4 is normal in H_2 and H_2 centralizes $H_4/\langle z \rangle$. We have $[H_1, H_4] \leq H_1 \cap H_2 = \langle z \rangle$ and so H_1 also centralizes $H_4/\langle z \rangle$ and H_4 is normal in H. Thus, H centralizes $H_4/\langle z \rangle$ and therefore there is no $h \in H$ inducing an outer automorphism on H_4 (because otherwise such an element h would act nontrivially on $H_4/\langle z \rangle$). It follows that $C_H(H_4)$ covers H/H_4. But H/H_4 is nonabelian since $H_4 \leq H_2$ and $H/H_2 \cong H_1/\langle z \rangle \cong D_8$. This contradicts Lemma 1.1. We have proved that H_2 must be abelian and so H_2 is either abelian of type $(4, 2, \ldots, 2)$ or elementary abelian. In any case, $N = Z_0H_2 = \langle z_0 \rangle \times H_2$ since $z_0 \in Z(G)$.

(β1) Suppose that H_2 is abelian of type $(4, 2, \ldots, 2)$, where $\mho_1(N) = \mho_1(H_2) = \langle z \rangle$. Set $E = \Omega_1(H_2)$ so that $\Omega_1(N) = \langle z_0 \rangle \times E$ and all elements in $H_2 - E$ are of order 4. Let h be an arbitrary element in $H_2 - E$ so that $h^2 = z$.

We consider first the possibility that H_1 is minimal nonabelian nonmetacyclic. Let x be any element of order 4 in H_1 so that $x^2 = z$ or $x^2 = zz_0$, where $\langle z_0 \rangle = H_1'$. Suppose that $[x,h] \neq 1$. By Lemma 1.2, $[x,h] = x^2h^2 = x^2z$. On the other hand, $[x,h] \leq H_1 \cap H_2 = \langle z \rangle$ and so $[x,h] = z$, which gives $x^2 = 1$, a contradiction. Hence $[x,h] = 1$ and, since H_1 is generated by its elements of order 4 (noting that $\Omega_1(H_1) \cong E_8$) and $H_2 = \langle H_2 - E \rangle$, we get $[H_1, H_2] = \{1\}$.

We apply Lemma 1.1 in the factor group $\bar{H} = H/\langle zz_0\rangle$. We have $\bar{H}_1 \cong D_8$ and $\overline{\langle H_1, h\rangle} = \bar{H}_1 * \langle \bar{h}\rangle$ is the central product of $\bar{H}_1$ with $\langle \bar{h}\rangle \cong C_4$, where $\bar{H}_1 \cap \langle \bar{h}\rangle = Z(\bar{H}_1)$, a contradiction.

We consider now the possibility where $H_1 = D \times \langle z\rangle$ with

$$D = \langle a, t | a^4 = t^2 = 1, a^t = a^{-1}\rangle, \quad a^2 = z_0, \quad \text{and} \quad \langle z_0\rangle = D'.$$

If $[a, h] \neq 1$, then $[a, h] \in H_1 \cap H_2 = \langle z\rangle$ and so $[a, h] = z$. On the other hand, Lemma 1.2 implies $[a, h] = a^2h^2 = z_0z$, a contradiction. Hence $[a, h] = 1$ and so a centralizes $H_2 = \langle H_2 - E\rangle$. It follows that $A = ZH_2 = \langle a\rangle \times H_2$ is an abelian maximal subgroup of H with $\exp(A) = 4$ and A is normal in G. We have $\Omega_1(H_1) = H_1$ and so $\Omega_1(H)$ contains the maximal subgroup H_1E of H, where $E = \Omega_1(H_2)$ with $|H_2 : E| = 2$. If $[t, h] = 1$, then $D = \langle a, t\rangle \cong D_8$ centralizes $\langle h\rangle \cong C_4$, contrary to Lemma 1.1. Hence $[t, h] \neq 1$ and, since $[t, h] \in H_1 \cap H_2 = \langle z\rangle$, we get $[t, h] = z$ with $z = h^2$. Thus, $\langle t, h\rangle \cong D_8$ and so th is an involution in $H - (H_1E)$. We have proved that $\Omega_1(H) = H$ and, since $\Omega_1(G) \leq H$, we get also $\Omega_1(G) = H$. Also, $H' = \langle z, z_0\rangle \cong E_4$ and therefore, by a result of A. Mann (Lemma 1.5), A is the unique abelian maximal subgroup of H. Take a subgroup G_1 of G with $H < G_1 < G$ and $|G_1 : H| = 2$. It follows that G_1 must be a Wilkens group of type (a) with $\Omega_1(G_1) = H$, since $|G_1 : H| = 2$ and A is the unique abelian maximal subgroup of H (with $\exp(A) > 2$). In that case G_1/A is cyclic. On the other hand, setting $S_1 = S \cap G_1$, we have $G_1 = HS_1$ and $S_1 \cap H = Z$. Thus, if $g \in S_1 - H$, then $g^2 \in Z \leq A$. This contradicts the fact that $G_1/A \cong C_4$.

(β2) We have proved that H_2 (and so also $N = \langle z_0\rangle \times H_2$) is elementary abelian. Suppose first that H_1 is minimal nonabelian nonmetacyclic. In this case z_0 is not a square in H_1, where $\langle z_0\rangle = H_1'$. There is $b \in H_1 - Z$ with $b^2 = z$, $t = ab$ is an involution, and $a^2 = zz_0$. Now, $A = ZN = \langle a\rangle N$ is normal in G and $(\langle t\rangle N)^s = \langle b\rangle N$ (recalling that $S = \langle s\rangle \times \langle z\rangle$) since $G/N \cong M_{2^n}, n \geq 4$, and so G acts nontrivially on the four-group H/N. The subgroup $\Omega_1(H)$ contains the maximal subgroup $\langle t\rangle N$ of H, where $\Omega_1(H_1) = \langle z, z_0, t\rangle \cong E_8$ and $N \cap H_1 = \langle z, z_0\rangle$. If t centralizes H_2, then $\langle t\rangle N$ is elementary abelian. But $(\langle t\rangle N)^s = \langle b\rangle N$ and $\langle b\rangle N$ is not elementary abelian, a contradiction. Hence t does not centralize H_2 and so $[H_2, t] = \langle z\rangle$ since $[H_2, t] \leq H_1 \cap H_2 = \langle z\rangle$. It follows that $\langle t\rangle N$ is nonabelian and so $\langle b\rangle N$ is also nonabelian. In particular, $H' = \langle z, z_0\rangle \cong E_4$, $[H_2, b] = \langle z\rangle$, and so $\langle b\rangle$ is normal in $H_2\langle b\rangle$. Let u be an involution in H_2 with $[b, u] = z$ so that $\langle b, u\rangle \cong D_8$ and therefore bu is an involution. But $bu \in H - (\langle t\rangle N)$ and so $\Omega_1(H) = H$. It follows that H must be a Wilkens group of type (a) or (b). In that case H must possess an abelian maximal subgroup M which is also unique (by a result of A. Mann since $|H'| = 4$). We have $M \geq H' = \langle z, z_0 \rangle$. If M does not contain H_2, then M covers H/H_2 which is nonabelian (since $H/H_2 \cong H_1/\langle z \rangle \cong D_8$), a contradiction. Hence $M \geq H_2$ and so $M \geq \langle H', H_2 \rangle = N$. Since $\langle t \rangle N$ and $\langle b \rangle N$ are nonabelian, we get that $M = A = \langle a \rangle N$ is abelian (of exponent 4). Thus H is a Wilkens group of type (a) with respect to A and so the involution t must invert each element in A. In particular, $a^t = a^{-1} = aa^2 = a(zz_0)$. This is a contradiction since $H_1' = \langle z_0 \rangle$.

It remains to investigate the case where $H_1 = D \times \langle z \rangle$ with

$$D = \langle a, t | a^4 = t^2 = 1, a^t = a^{-1} \rangle, \quad a^2 = z_0, \quad \langle z_0 \rangle = D', \qquad \text{and}$$
$$S = \langle s \rangle \times \langle z \rangle, \quad \langle s \rangle \cap H = \langle a \rangle, \quad (\langle t \rangle N)^s = \langle at \rangle N, \quad \text{since } G/N \cong M_{2^n}, n \geq 4.$$

Obviously, in this case $\Omega_1(H) = H = \Omega_1(G)$. Suppose that t does not centralize H_2. Then $[t, H_2] \leq H_1 \cap H_2 = \langle z \rangle$ and so $[t, H_2] = \langle z \rangle$ and $H' = \langle z, z_0 \rangle \cong E_4$. Then $\langle t \rangle N$ and $\langle at \rangle N = (\langle t \rangle N)^s$ are nonabelian. But H is a Wilkens group with $\Omega_1(H) = H$ and so H must have an abelian maximal subgroup U which is unique (by a result of A. Mann). If U does not contain H_2, U covers H/H_2 and this is a contradiction, since $H/H_2 \cong D_8$. Hence $U \geq \langle H', H_2 \rangle = N$ and so $U = A = \langle a \rangle N$ is abelian of exponent 4. Thus, H is a Wilkens group of type (a) with respect to A. In particular, the involution t inverts each element in A and so t centralizes H_2, a contradiction. Hence t centralizes H_2 and so $\langle t \rangle N$ is elementary abelian. In that case $\langle at \rangle N = (\langle t \rangle N)^s$ is also elementary abelian. In particular, $D = \langle t, at \rangle \cong D_8$ centralizes H_2 and so $H = D \times H_2$. It follows that G is a Wilkens group of type (c), a contradiction. Our statement (iv) is proved.

We have proved that the nonmodular factor group $G/\langle z \rangle$ ($z \in \Omega_1(Z(G))$) (according to our statement (i)) is not isomorphic to any Wilkens group (according to (ii), (iii), and (iv)). This is a final contradiction and so the Main Theorem is proved.

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